The qualitative theory of elementary transformations of one- and multichannel quantum systems in the inverse-problem approach. The construction of transformations with given spectral parameters

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New results on simple, universal algorithms of spectrum, scattering, and decay control are reviewed. The limiting (most informative) cases are studied, such as the separation in space and in energy of seemingly inseparable states close in energy and in the shape of the moduli of their wave functions, and conversely, decrease of the level spacing to the point where the levels become degenerate. The enrichment of the set of elementary transformations in going to multichannel systems (to a vector structure of the spectral parameters, the levers for controlling coupled channels) is demonstrated. For example, it becomes possible to concentrate waves in a single channel by "pumping" them out of other channels. It becomes clear that soliton-like potential wells are the "carriers" of not only one-channel, but also partial channel states when they are separated from the other states. A reflectionless interaction which has no analog in the one-channel case is described. The "paradoxical" coexistence of confinement and transparency at the same energy for the same interaction is discussed. [S1063-7796(99)00102-3]

INTRODUCTION

Only from the simplest and most accessible objects can the most valuable truths be derived........
By intuition I mean not belief in the shaky evidence of the emotions or the deceptive judgment of the disordered imagination, but the understanding of a clear and alert mind which is so direct and striking, that it leaves no doubt...

René Descartes

Many new developments have occurred since our third review in the series "Lessons in quantum intuition" in the Journal of Particles and Nuclei,[1] and indeed our earlier views have changed somewhat.

The essence of quantum theory is the establishment of relations between interaction characteristics and observables—spectral data. Learning how to change observables at will by the suitable variation of an interaction is an excellent way to understand the rules governing the microscopic world and the processes occurring in it. The mathematical formalism of the inverse problem and supersymmetry[1)] (SUSYQ) can help in doing this.[2–7] In fact, in the inverse problem the initial data are the spectral characteristics. This makes the formalism convenient for studying the elementary transformations of the interaction resulting from the variation of individual observable parameters. Here the inverse problem is not the traditional special numerical technique for the approximate reconstruction of the potential of a particular system on the basis of measured (with errors) scattering and spectral data. The inverse problem that we are discussing is broader: the revelation of the general laws by which quantum systems are constructed, without restricting ourselves to examples realized in nature, which form only a set of measure zero of all the possible objects. In the world in which we live they represent only rare points in a continuum of possibilities. We are now learning how, at least in theory, to transform objects of different nature into each other. This gives us a deeper, more unified and simpler, understanding of these objects, as we build bridges of continuous transitions connecting them with each other (a continuum of "trajectories" describing the evolution of interactions and wave functions; see our books and reviews[8]).

One example is the transformation of free motion into a system with a potential well possessing bound states whose locations are *a priori* specified: a potential whose low-lying levels are oscillator or square-well levels, or the levels of a linear well; see Fig. 5.2 in Ref. 8. Another is the "sweeping up" of a scattering state to form a bound state buried in the continuum; see the figures in Refs. 8, 6, and 9. A further example is the creation of new and the annihilation of existing bound-state energy levels, and so on.[8]

Every quantum system has its own, so to speak, "passport": its spectral function. This function indicates the energies at which physical states exist (a discrete, or continuous, or band spectrum) and associates with each such energy a spectral weight factor (SWF) c_ν or M_ν, related to the spatial distribution of the waves (their localization or "registration").

The qualitative theory of "quantum design" for one-dimensional and one-channel systems is nearly complete: we understand which potential transformations are needed to obtain the desired changes of physical properties. The elementary components, the "building blocks," of these transformations have been determined.[1,8–11] The problem now is to extend these achievements to objects with more complicated structure.

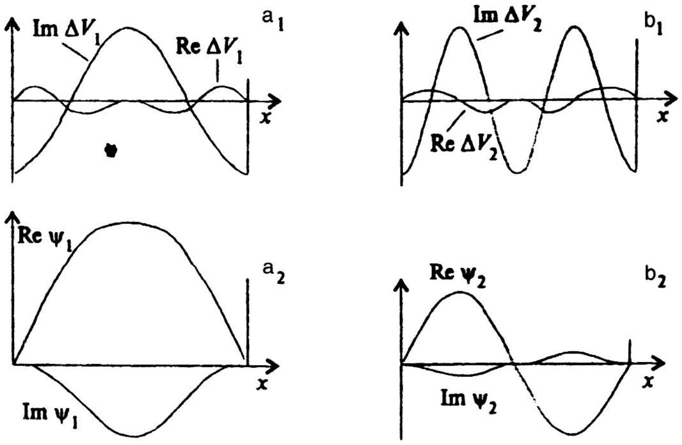

FIG. 1. The transformation $\Delta V(x)$ of the infinite square-well potential and wave functions Ψ_n when an imaginary unit is added to one of the bound-state levels $\mathring{E}_n \to E_n + i$ (a shift in the "imaginary" direction) without changing the spectral parameters of the other states. The shift is effected mainly by imaginary additions to the potential in accordance with the qualitative rules for real shifts. (a) Shift of the ground state by $+i$: $\mathring{E}_1 \to E_1 + i$; (b) shift of the second level by $+i$: $\mathring{E}_2 \to E_2 + i$.

In the first part of this review we deal with one-channel systems, and in the second part we study multichannel systems, emphasizing their special features compared to the former.

ONE-CHANNEL PROBLEMS

Imaginary additions to the energy levels

Let us consider the transformation of individual bound states into decaying states in the one-dimensional case.

It turns out that the usual rule, which we formulated earlier,[6,8] works in this case, but now we shall apply it to the imaginary part of the potential perturbation. A single, the nth, bound-state energy level is shifted upward (downward), while the infinite number of other levels remain fixed, by adding n repulsive bumps (attractive little wells) $\mathrm{Im}\,\Delta V(x)$ to the original potential $V(x)$ in the regions where the nth wave function $\psi_n(x)$ is most sensitive to potential perturbations, i.e., at the centers of its n antinodes. To keep the other levels from shifting, $n+1$ compensating little wells (bumps) are added near the nodes of $\psi_n(x)$. Examples of perturbations of potentials and eigenfunctions when an imaginary unit is added to a selected level are shown in Fig. 1 for an infinite square well with $\mathring{E}_1 \to E_1 = \mathring{E}_1 + i$, $\mathring{E}_{j\neq 1} = E_{j\neq 1}$ and $\mathring{E}_2 \to E_2 = \mathring{E}_2 + i$, $\mathring{E}_{j\neq 2} = E_{j\neq 2}$. The real part of the potential perturbation is relatively small. We suggest that the reader try to understand its shape (so far the authors have not managed to do this).

How to isolate one state from a multiplet of nearly degenerate states

Let us recall the usual rule. Perturbations $\Delta V(x)$ for shifting one bound state in space are constructed from universal elementary blocks: combinations of one repulsive bump (barrier) and one little well for each antinode of the standing wave. For the nth state, n such elementary building blocks are needed: n (barrier+little well) pairs. Here the location of the levels on the energy scale does not change, owing to the mutual cancellation of the attraction and repulsion. It turns out that the smallest values $\inf[|\psi_n(x)|^2]$ of the selected state remain fixed in space. At these points the modulus of the perturbation also takes its smallest values. It has been shown that for a shift in x of a single state without changing the spectral parameters of the others, the latter undergo a sort of "recoil" (they slightly shrink back in the opposite direction, which tends to increase their separation from the selected state). In the limit, the selected state can thus be taken to infinity, which is equivalent to annihilating the corresponding energy level in the spectrum of the remaining system.[8] Conversely, a new level can be produced by attracting a state from infinity.

Let us now see how this simple rule works in another instructive limiting case, that of very closely spaced levels, in order to enrich our qualitative understanding of the properties of quantum systems. When the structure of the maxima (antinodes) and minima of $|\psi_n(x)|^2$ of nearly degenerate states is barely distinguishable, it is not clear what to "grab hold of" in order to separate one of these states in space. A contradiction seems to arise: earlier, we shifted different states with different numbers of antinodes of the standing waves by using different numbers of (barrier+little well) blocks. But quasidegenerate states (in original systems which have many wells) have the same number of antinodes. Each such antinode is similar to the corresponding antinodes of the other states up to a sign (Fig. 2). A shift of any of the states of a multiplet which are similar to each other requires poten-

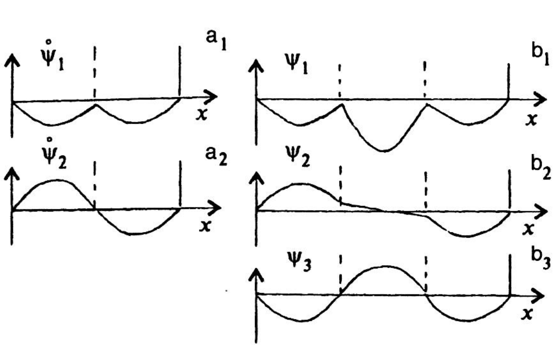

FIG. 2. The unperturbed states in an infinite square well partitioned by weakly transmissive δ barriers (dashed lines), which causes states to be grouped into nearly degenerate multiplets: $a_{1,2}$ is the doublet of ground and first excited states for the double well, and $b_{1,2,3}$ is the triplet for the triple well. Inside the partial wells the unperturbed wave functions are similar for all the bound states up to a sign. Only the second state of the triplet, which has a node at the center of the middle well and takes small values on this segment, has "proportionality factor" close to zero relative to the other states.

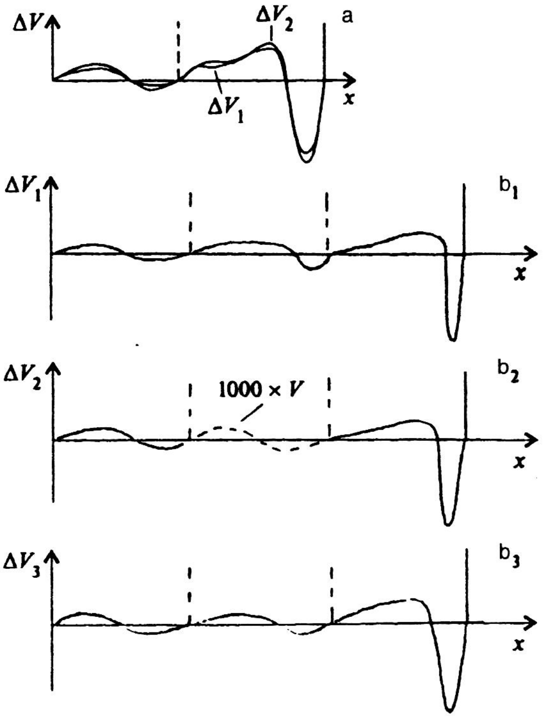

FIG. 3. The potentials ΔV_i, transformed from the original square well with δ-barrier partitions (vertical dashed lines), shifting only the selected ith state of the lower multiplet toward the right-hand wall of the potential: (a) in a two-well system; (b) in a three-well system. The partial sections of these potentials are the elementary blocks (barrier–little well). The small central part ΔV_2 in Fig. 3 b_2 is multiplied by 1000 to make it more visible.

tial perturbations similar in shape. In Ref. 14 we showed that because of this, by using very similar transformations of the original potential it is possible to shift away (in x) only one of a group of states very close in energy and in shape, while the other standing waves of the eigenfunctions are shifted even in the opposite (!) direction, although the same behavior for all the terms of the multiplet might have been expected for similar transformations of the potential.

Using exactly solvable models with two and three wells, we shall show that the "paradox" admits a simple explanation. In separating a single state, we discovered two types of mechanism for shifting standing waves. There is a shift inside each partial well. This corresponds to our earlier intuition: each antinode acquires a little bump with a little well in the potential perturbation (Fig. 3). This perturbation actually transforms the individual antinodes, sweeping them to one side for all the states of the multiplet. However, on this transformation of the functions there is superimposed a *change of the amplitudes* of the individual antinodes. For the selected state these amplitudes grow in the direction of the desired shift, enhancing the shifts of the individual antinodes, while for the other states the amplitudes grow in the opposite direction, overpowering the weaker partial shifts at the ends of each antinode (Fig. 4). This different "amplitude modulation" occurs owing to the high sensitivity to the small difference between the energies of the states of the multiplet—it is a *resonance enhancement* of the partial antinodes: for different states the resonance conditions are satisfied for different parts of the function to different degrees, which leads to a redistribution of the wave intensity in them. The same phenomenon has been discovered in another limiting case: that where a scattering state is "swept up" to form a bound state.

Two wells

As the initial, exactly solvable model we take an infinite square well of width $2a$, divided into two parts by a weakly transmissive barrier $V\delta(x-a)$. The values $V=-2\cot(a)$ and $a=\pi-0.1$ are chosen so that the second state has a node at the location of the δ barrier and the wave function is continuous there (see Fig. 2). The wave functions of the two lowest, nearly degenerate bound states have the form

$$\mathring{\varphi}_1^l=-\sin(x);\quad \mathring{\varphi}_1^r(x)=-\sin(2a-x); \tag{1}$$

$$\mathring{\varphi}_2(x)=\frac{a}{\pi}\sin\left(\frac{\pi}{a}x\right), \tag{2}$$

where the functions on the left- and right-hand sides of the full potential well are labeled l and r, and the wells are continued (anti)symmetrically to the other part; see Fig. 2a. They are normalized so that the derivative of the function $\mathring{\varphi}_{1,2}^r(x)$ at the point $x=2a$ is equal to 1 (a special form of "regular" solutions). The derivative at this point for the νth bound state $\mathring{\Psi}-\nu(x)$ with the standard normalization can be chosen as the spectral weight factor (SWF).

In Fig. 3a we show the potential transformations shifting the ground state or a higher state to the right.

These transformations are similar, but they affect the two lowest states barely distinguishable in energy differently (Figs. 4a and 4b). However, there are several common features. All the standing waves inside an individual (left or right) well are shifted to the right in accordance with our ideas developed earlier, but the different amplitude modulation of the antinodes of the standing waves ensures an overall shift of the two states in opposite directions. This occurs as a result of the interference of multiply reflected waves (resonance sensitivity to the perturbation): In one part of the well this interference is constructive and tends to enhance the wave, while in the other it is destructive and tends to weaken it.

Three wells

Let us consider the case of an original infinite square well of width $3a$ with two weakly transmissive partitioning barriers $V\delta(x-a)$ and $V\delta(x-2a)$, causing the three lowest bound states to be nearly degenerate (see Fig. 2b).

The values of V and $a=\pi-0.1$ are chosen so that the third state has nodes at the δ barriers and the wave function at these points is continuous (the discontinuity of the function due to the δ barrier is proportional to the value of the function at the barrier point and vanishes if the barrier is located at a node of the function).

The unperturbed function of the second state $\mathring{\varphi}_2(x)$ has an interesting feature. It would appear that contradictory requirements are imposed on it. On the one hand, this function must have a node at the center of the well ($x=3a/2$), but, being a member of a multiplet of quasidegenerate states, inside the individual middle well it must be approximately

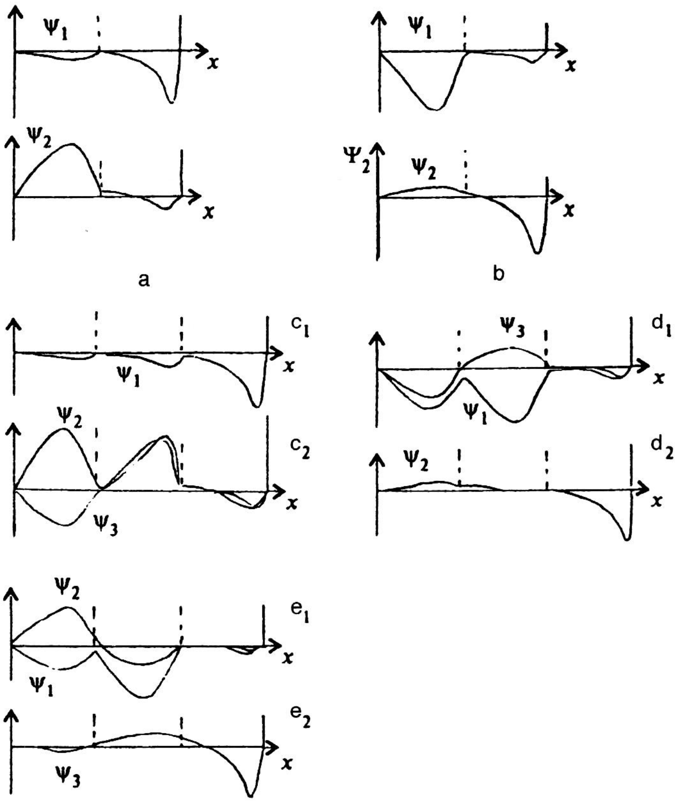

FIG. 4. The wave functions (un-normalized) corresponding to the potentials shown in Fig. 3 (the potential partitions are shown by the dashed lines): (a) ground state of the doublet shifted toward the right-hand wall of the potential, second state shifted to the left, where (b) the second state of the doublet is localized at the right-hand wall and the first state is shifted to the left; (c_1) ground state of the triplet shifted to the right, where (c_2) the second and third states of the triplet shrink back to the left; (d_1) ground and third states of the triplet shrink back to the left, while (d_2) the second state of the triplet is shifted to the right; (e_1) ground and second states of the triplet shrink back to the left, while (e_2) the third state of the triplet is shifted to the right. Inside the partial wells the wave functions are usually shifted to the right, but the resonance modulation of the amplitudes isolates the right-hand well only for the selected state, while for the others it leads to an overall shift (shrinking back) in the opposite direction.

proportional to the ground state and the third state, which have antinodes rather than nodes at this point. The only way to force the function $\mathring{\varphi}_2(x)$ to satisfy both requirements is to make its value small in the central well, which can approximately be interpreted as "proportionality, with proportionality factor close to zero" to the functions of the adjacent states $\mathring{\varphi}_{1,3}(x)$ (see Fig. 2b).

The transformed potentials and wave functions are determined by the same expressions as in the two-well case.[14] In Fig. 3b we show the changes of the square-well potential divided into three parts by two δ-function barriers when only one of the SWFs is increased (the corresponding transformed wave functions are shown in Figs. 4c–4e):

(1) a shift of the ground state to the right-hand wall of the potential (Fig. 4c);
(2) a shift of the second state to the right-hand wall (Fig. 4d);
(3) a shift of the third state to the right-hand wall (Fig. 4e).

Comparison of the results for the three- and two-well systems confirms these conclusions.

The idea of the effect of the resonance mechanism on wave localization which we have encountered here should also be useful for explaining the transformations of continuum states, to which we now turn.

The transformation of scattering states into bound states buried in the continuum

Bound states buried in the continuum (BSCs) were studied by us earlier.[6,8,12] However, now, on the basis of the above discussion, their appearance can be explained as the sweeping up of the function at one point of the continuum $E=E_b$ toward the coordinate origin.[13] It is also possible to give the interpretation of a continuum originating with the adjacent states; see the example in Fig. 5a, where we show the wave function for a scattering state with energy close to

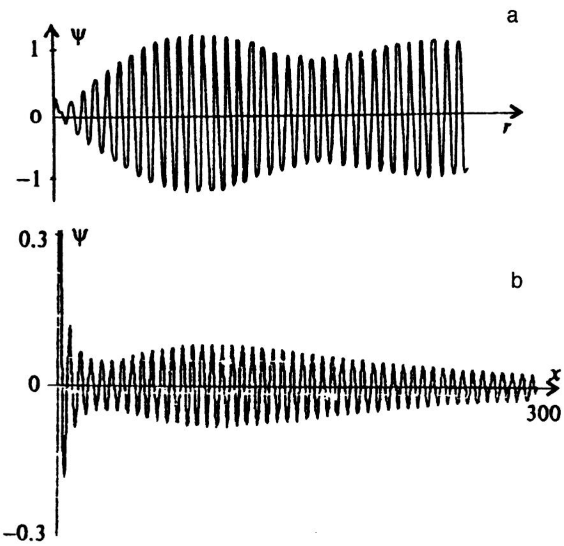

FIG. 5. (a) Beats in the wave function of a scattering state close to a bound state buried in the continuum (BSC). The beats are due to local periodic mismatching of the oscillation phases of the potential and the wave function (in contrast to BSCs). The sweeping up to the left occurs only on individual segments, while on adjacent ones it even occurs in the opposite direction, which leads to local enhancements of the standing-wave amplitude. At large distances the beats weaken and the function reaches the regime of unperturbed oscillations without a phase shift (the nodes of the function are not shifted in the transformation). (b) Breakup of the BSC wave function into two parts as the energy levels of two BSCs approach each other.

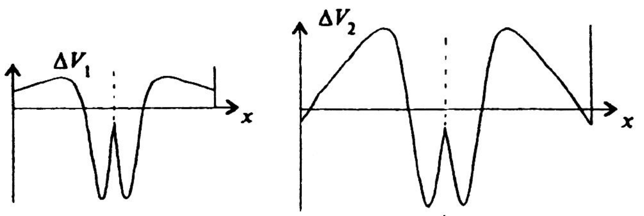

FIG. 6. Change of the depth of a square well with weakly transmissive partition at the center in the splitting of the lower doublet of bound states E_1, E_2: (a) the addition to the potential shifting the ground state E_1 downward; (b) the perturbation of the potential shifting the second state E_2 upward.

E_b. Whereas for the BSCs a little well–barrier block corresponds to each antinode of the wave function, for the adjacent states the nodes of the function not only will not exactly coincide with the ends of the blocks (weakening of the sweeping), but in individual segments strong dephasing will occur, which can lead to (local) sweeping of the waves in the opposite direction, thus producing the beats seen in Fig. 5a. There is also the competition between the two trends of sweeping and modulation of the oscillation amplitudes of the standing waves due to the constructive or destructive interference of waves on different segments of the x axis. The BSCs themselves can be viewed as a special case of this picture, where a wave is swept to the coordinate origin along the *entire* axis, and there is only *one* antinode of the beat.

The mutual approach of two BSC levels leads to splitting of the BSCs into two groups of waves. One stays near the coordinate origin. The other moves away from it (in the limit it goes to infinity); see Fig. 5b, which is reminiscent of the breakup of ordinary bound states when they become degenerate.[17] Here we see beats in the potential due to the "addition" of oscillations with similar frequencies (the amplitudes fall off as $1/x$).

Breakup of a doublet

The breakup effect and the *spatial repulsion* of states as their energy levels approach each other was discovered in Ref. 17. Let us now consider the reverse process in which two closely spaced levels of a multiplet move away from each other. In Figs. 6a and 6b we show the changes of the flat bottom of a square well with a weakly transmissive barrier in the middle as the ground-state level of the lower doublet is lowered and the second state is shifted upward. The other levels remain fixed. The two perturbations $\Delta V_{1,2}(x)$ are formed of blocks (barrier–little well, little well–barrier) which *shift* the antinodes of the wave functions toward each other.

Inversion of the original potential by a SUSYQ (Darboux) transformation

A remarkable property of SUSYQ is that by factorizing the original Hamiltonian H, a second-order differential operator, into first-order differential operators A^+ and A^-, and then interchanging the $A^{\pm}$, we obtain *very simple* expressions for the potential and wave functions of the new Hamiltonian in terms of V and ψ of the original system.

Inversion of individual delta potentials

Let us consider the transformation of the δ potential $V_0(x)=v_0\delta(x)$ in the creation of a bound state with preservation of the reflection properties of the system. We shall apply a SUSYQ (Darboux) transformation, which *preserves* the spectral structure, except for the single bound-state level added to or subtracted from the spectrum of the system in this transformation.[18,21,23] We write the original Hamiltonian in factorized form:

$$H_-=\frac{d^2}{dx^2}+V_0(x)=A^+A^-+\epsilon,\quad \epsilon<0, \tag{3}$$

where ϵ is the so-called factorization energy and A^- has the form

$$A^-=-\frac{d}{dx}+W(x), \tag{4}$$

A^+ being its Hermitian conjugate. We can find $W(x)$ from the equation

$$A^-\psi^-=\left[-\frac{d}{dx}+W(x)\right]\psi^-=0. \tag{5}$$

Here ψ^- is a solution of the Schrödinger equation for the Hamiltonian H at energy ϵ and

$$A^-=-\frac{d}{dx}+\frac{d}{dx}\ln\psi^-. \tag{6}$$

For the SUSYQ partner $H_1\equiv A^-A^++\epsilon$ we have the expression (the proper Darboux transformation)

$$H_1=-\frac{d^2}{dx^2}+V_0-2\frac{d^2}{dx^2}\ln\psi^-. \tag{7}$$

Let ψ_E^- be an eigenfunction of H at energy E. Then $\psi_E^+\equiv A^-\psi_E^-$ is a solution of the Schrödinger equation with H_1 at the same energy:

$$H_1(A^-\psi_E^-)=A^-A^+(A^-\psi_E^-)+\epsilon(A^-\psi_E^-)=A^-H\psi_E^-$$
$$=EA^-\psi_E^-. \tag{8}$$

At the factorization energy ϵ the function ψ^+ is obtained by solving the first-order differential equation

$$A^+\psi^+=\left[\frac{d}{dx}+\frac{d}{dx}\ln\psi-\right]\psi^+=0, \tag{9}$$

which gives

$$\psi^+=\frac{1}{\psi^-}. \tag{10}$$

The solution $\psi^-(x)$ itself can be constructed as an arbitrary combination of linearly independent solutions of the original Schrödinger equation $\psi_1^-(x)$ and $\psi_2^-(x)$ at factorization energy ϵ:

$$\psi^-(x)=\psi_1^-(x)+c\psi_2^-(x). \tag{11}$$

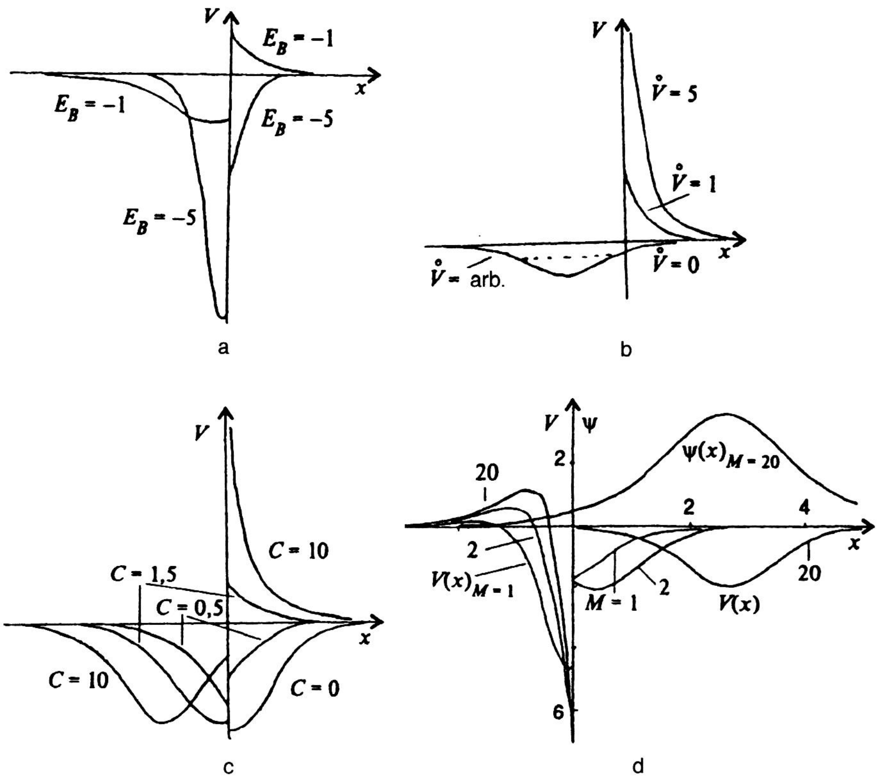

FIG. 7. The transformation of the original δ-like potential peak in the generation of a bound state, depending on (a) the binding energy of the generated state, (b) the strength of the potential barrier $\overset{\circ}{V}$, and (c) the spectral parameter c determining the spatial location of the bound state. (d) The potentials in the double SUSYQ transformation without inversion (generation of a bound state and a spatial shift).

In particular, $\psi^-(x)$ can be chosen to be nodeless and asymptotically growing. Then, owing to (10), $\psi^+(x)$ will be a normalizable (asymptotically damped) solution of the new Schrödinger equation, which implies the creation of a new bound state.

From the expression for the transformed potential we can clearly see the effect of inversion of the potential peak (well). In fact,

$$V_1(x) = V_0(x) - 2\frac{d^2}{dx^2}\ln\psi^-(x)$$
$$= -V_0(x) + 2\epsilon + 2\left\{\frac{[\psi^-(x)]'}{\psi^-(x)}\right\}^2, \qquad (12)$$

where we have used the expression for $[\psi^-(x)]''$ following directly from the Schrödinger equation. It should be noted that the term V_0 appears with the minus sign in the last equation. In our case V_0 is a δ function, and so the other (finite for all x) terms cannot compensate the flip of the δ barrier (well):

$$V_1(x) = -v_0\delta(x) + 2\epsilon + 2\left\{\frac{[\psi^-(x)]'}{\psi^-(x)}\right\}^2. \qquad (13)$$

It should be stressed that Eq. (12) is true also for any one-dimensional potential. However, the change of sign may be hidden in the background of additional terms.

This transformation does not change the modulus of the reflection and transmission coefficients. In fact, let $\psi_E^-(x)$ be a solution of the original Schrödinger equation at energy E satisfying the asymptotic condition

$$\psi_E^-(x) \sim \exp(-ikx) + r(k)\exp(ikx), \quad x \to \infty.$$

Then according to (6) and (8) we obtain the expression for the asymptote of the perturbed solution:

$$\psi_E^+(x) = [\psi_E^-(x)]' + [\psi_\epsilon^-(x)]'\psi_E^-(x)/\psi_\epsilon^-(x),$$

where

$$\psi_\epsilon^-(x) \sim \exp(\kappa x), \quad x \to \infty,$$

and

$$\psi_E^-(x) \sim \exp(ikx) + \frac{\kappa - ik}{\kappa + ik}r(k)\exp(ikx), \quad x \to \infty.$$

Here the solution is normalized to unit amplitude of the incident wave. Therefore, the reflection coefficient $r(k)$ remains unchanged in modulus, and only its phase changes.

In Fig. 7 we show how the original δ peak is not only inverted (to form a δ well), but acquires potential perturbations in the creation of a bound state, depending on the binding energy of the created state, from the barrier strength $\overset{\circ}{V}$ and from the spectral parameter c determining the location of the bound state in space. We also show the potential under a double SUSY transformation, when the δ peak returns to its original state after two inversions (Fig. 7d).

Transparent perturbations of periodic potentials

The inverse problem for periodic potentials has been studied by a number of authors; see the literature cited in Ref. 8. Periodic potentials have a characteristic band spectrum with transmission zones (allowed zones) in which Bloch waves propagate. The solutions in the forbidden zones (gaps) grow exponentially (they "get out"). Let us create a bound state in the lowest forbidden zone of a periodic poten-

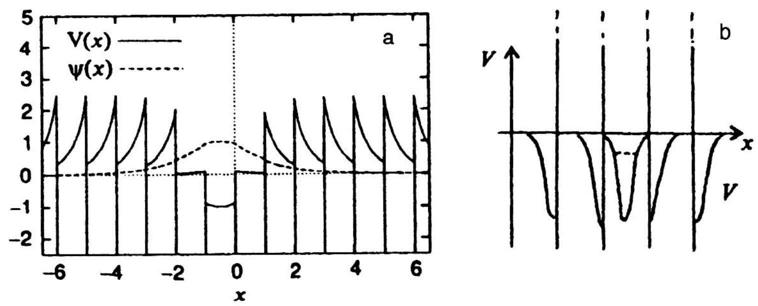

FIG. 8. The SUSYQ-generated transformation of (a) the Dirac comb in the generation of a bound state in the lower forbidden zone, accompanied by inversion of the δ peaks and the appearance of compensating finite potential peaks for preserving the spectral parameters of the system; (b) periodic δ wells into a Dirac comb and the compensating potential wells of finite depth.

tial, while preserving the zone structure. However, in this case purely soliton-like potential perturbations cannot be transparent, as noted above, since local lowering of the lower boundaries of forbidden zones to allowed zones occurs. These depressions act as effective potential barriers "hanging from above" and give strong reflection.[8]

Let us consider two different initial periodic potentials:[15] the so-called Dirac comb, a sum of periodically located δ barriers (or wells)

$$V_0(x)=\sum_{n=-\infty}^{\infty} v_0\delta(x-na), \tag{14}$$

and a periodic potential of the form

$$V_0(x)=v_n\cos(\pi x)^{2n}, \tag{15}$$

where n is an integer and v_n is chosen to normalize the potential so that in the $n\to\infty$ limit the potential becomes a Dirac comb with $v_0=1$.

To generate a bound state in the lower forbidden zone of a periodic potential, we use a Darboux (SUSYQ) transformation which does not spoil the characteristics of the rest of the spectrum. In fact, if the energy E belongs to an allowed zone, i.e., if the asymptote of ψ_E^- does not grow exponentially, the new solution $A^-\psi_E^-$ also will not diverge, because the function $W(x)$ in the operator A^- is finite at large $|x|$. In other words, the Darboux transformation does not shift the boundaries of the energy gaps, even though the transformed potential is no longer periodic.

In Fig. 8a we show the effects of a SUSYQ transformation generating a bound state in the lower forbidden zone for a Dirac comb consisting of potential δ barriers (peaks). The potential strength is taken to be $v_0=1$, and the bound-state energy is chosen as $\epsilon=-1$ in dimensionless units.

Here the δ barriers become δ wells. The sharp finite peaks in the potential appear in order to raise the spectral bands, lowered when the comb was inverted, to their former places. The small well between the middle peaks ensures the appearance of a bound state.

The transformation of periodic δ wells with $v_0=-0.5$ is shown in Fig. 8b for bound-state energy $\epsilon=-1$. The finite potential wells which appear lower the zones, raised when the δ wells were flipped, to their former places, and the small well at the center ensures the appearance of the bound state.

In Fig. 9 we show the case of periodic potentials specified by (15) with different values of n. In Fig. 9a we show the transformation of the original potential with $n=1$ and $\epsilon=-1$. Here again the potential is flipped, although this is not as clearly seen (there is a "phase" shift of the original periodic potential). The results for $n=8$ and $n=32$ are

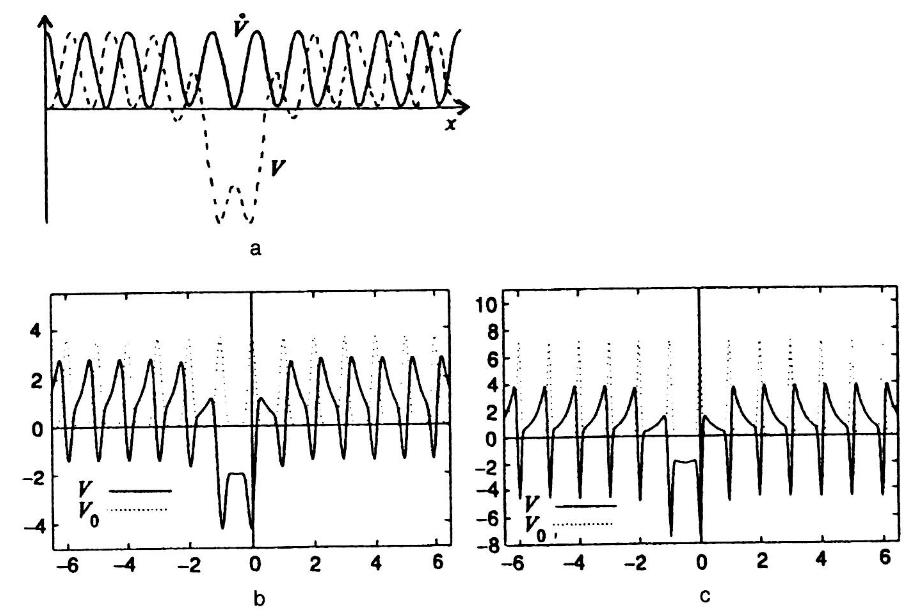

FIG. 9. The transformation of the smooth periodic potential (15) occurring in the generation of a bound state in the lower forbidden zone as a function of the parameter governing its approach to the Dirac comb: (a) $n=1$; (b) $n=8$; (c) $n=32$. The wave functions of the corresponding bound states are shown by a broken line.

shown in Figs. 9b and 9c. Increasing n causes the potential to approach the Dirac comb; cf. Fig. 8a.

In Figs. 7–9 we see the elements of the transformation of the potential ΔV which sweep the lower states toward the center (as in fig. 6) when a bound-state level is decoupled from the continuous spectrum. This tendency, which is opposite to the breakup and separation of individual parts of the states when the levels come together,[17] can also be seen in Fig. 5b.

Incidentally, we note that bound states can be generated below any of the allowed zones also in the approach of the ordinary direct problem for an arbitrarily small attractive perturbation of the potential, as occurs in the case of free motion along the entire axis. Similarly, it can be expected that a bound state will appear *above* each allowed zone for an arbitrarily small repulsive addition to the potential which thereby creates ''flipped potential wells'' embedded inside forbidden zones.

Up to now we have changed only individual spectral parameters. However, it turns out that by varying individual ''unphysical'' spectral characteristics it is possible to change an infinite (!) number of physical parameters within the framework of exactly (!) solvable models of the inverse problem and SUSYQ, which enriches our arsenal of quantum design algorithms.

New degrees of freedom in the spectral control of physical systems (the variation of "unphysical" spectral parameters)

A given Schrödinger operator corresponds to definite boundary conditions. These conditions can be imposed in accordance with physical requirements. However, there is still an infinite set of (''mathematical'') means of specifying the behavior of the solutions at the ends (asymptotes) of the region of wave motion. And in each such case there is a set of spectral parameters which uniquely fix the form of the potential $V(x)$. Variation of the individual mathematical parameters leads to a change of the potential which, as a rule, can be obtained only by changing an infinite number of physical parameters. However, the two-spectrum theorem[2)] (for example, where one spectrum is physical and the other unphysical; see Ref. 6 and references therein) states that two sets of eigenvalues without weight factors also fix the form of $V(x)$. Here a change of one eigenvalue in the two sets with no change in the locations of the others (which is described by exact models[31]) leads to a change of all (!) the weight factors (in this formulation of the problem they are not fixed). Unphysical states can also be generated by using SUSYQ (Darboux) transformations.[18,32] Therefore, the class of exactly solvable quantum models, which are of great value since they are convenient to use, is greatly extended.

An interesting example is the case of symmetry transformations of the oscillator potential $V(x)$, when the bound-state levels on the entire x axis (a single spectrum) simultaneously represent *two* spectra of the problem on the semiaxis. States whose wave functions do not vanish at the center of the well are unphysical states for half of the potential (on the semiaxis).

The creation of un-normalizable states

Let us consider the creation of unphysical ''levels'' in the case of initial δ barriers (or wells) $V\delta(x)$ located at the point $x=0$:

$$-\psi''(x)+V\delta(x)\psi(x)=E\psi(x). \tag{16}$$

We shall choose an energy below the physical spectrum $E=\epsilon$ as the energy of ''factorization'' of the Hamiltonian (3) into simpler first-order operators A^+, A^- [see (5) and (9)], where

$$\psi(x\leqslant 0)=\exp(\kappa_1 x),$$
$$\psi(x\geqslant 0)=b_1\exp(-\kappa_1 x)+c_1\exp(\kappa_1 x),$$
$$\kappa_1=\sqrt{-\epsilon}. \tag{17}$$

We shall interpret the SUSYQ transformation (see the details of the formalism in Ref. 18) as the generation of the unphysical state

$$\psi_{\rm unphys}(\epsilon,x)=\frac{1}{\psi(\epsilon,x)}, \tag{18}$$

i.e.,

$$\psi_{\rm unphys}(x\leqslant 0)=\exp(-\kappa_1(x)),$$

$$\psi_{\rm unphys}(x\geqslant 0)=\frac{1}{b_1\exp(-\kappa_1 x)+c_1\exp(\kappa_1 x)}, \tag{19}$$

where

$$b_1=\frac{V_0}{2\kappa_1},\quad c_1=1-\frac{V_0}{1\kappa_1}. \tag{20}$$

The potential in the transformed Hamiltonian (12) is shown in Fig. 10 (the sign flip of the original δ potential should be noted).

The closer the energy of the unphysical state is to that of a physical state, the more similar are their wave functions as solutions of the same Schrödinger equation with the same potential and with identical boundary conditions on one side at nearly the same energy. The physical state plus the part of the unphysical function close to it are pushed by the special auxiliary ''carrier'' well off to the side of the region of primary localization of the original physical state.

The behavior of the shape of the potential as the energy of the generated unphysical state approaches that of a physical state is reminiscent of the ''annihilation''[32] occurring when two physical states are degenerate.[3)] In the limit $\epsilon\to E_b$ the physical state is taken off to infinity, effectively vanishing from the spectrum of the potential, leaving the original δ well in place.

In Fig. 11 we show the transformation of the oscillator well in the generation of an unphysical state below the physical ground state $E_{\rm unphys}\to E_1$. Small transformations of this type were first demonstrated by Sukumar,[18] who, however, did not interpret the resulting picture as the creation of an unphysical state, and did not as clearly demonstrate the departure of the common part of the unphysical and physical states arbitrarily far away in the limit $E_{\rm unphys}\to E_1$.

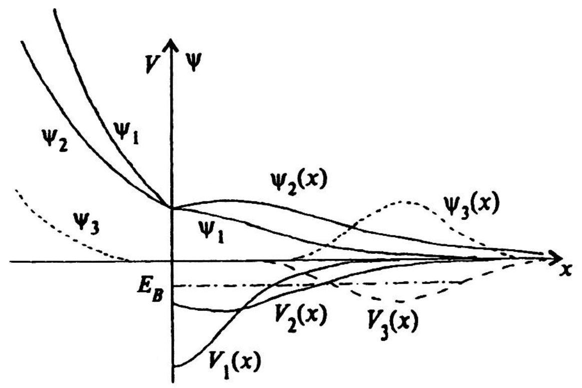

FIG. 10. The deformation of a δ-like potential well: $-V\delta(x)\to V\delta(x)+V_1(x)\to V\delta(x)+V_2(x)\to V\delta(x)+V_3(x)$ in the generation of an unnormalizable (unphysical) state with $E=\epsilon$ closer and closer to the single bound state with $E=E_b$. Homogeneous boundary conditions for the unphysical state were chosen at both asymptotes, so that the solution would be a wave purely decreasing to the right as $\exp(-\kappa_1 x)$ [the coefficients of $\exp(\kappa_1 x)$ were chosen to be zero]. The closer E is to E_b, the farther *to the right* is moved the well in which the bound state is predominantly localized. This *carrier-well* of the bound state gradually acquires a soliton-like shape as $E=\epsilon$ approaches E_b and then practically does not change. In the well, the physical $\psi_b(x)$ and unphysical $\psi_{1\to 2\to 3}(x)$ states on the right become more and more similar [on the left, $\psi_{1,2,3}(x)$ deviates from $\psi_b(x)$, growing exponentially]. The reflection (transmission) coefficients for continuum states remain unchanged under such SUSYQ transformations. Cf. the results of Ref. 8, where we first demonstrated the *inversion of the sign of the original δ wells and peaks* under a SUSYQ transformation.

A similar picture is also obtained for an original finite square well of depth $\mathring{V}$; see Fig. 12. It follows from these figures that under these transformations, in the limit the original symmetric well is squeezed, while its bottom is raised without breaking the symmetry (if we neglect the carrier potential which shifts the bound state). Obviously, it is

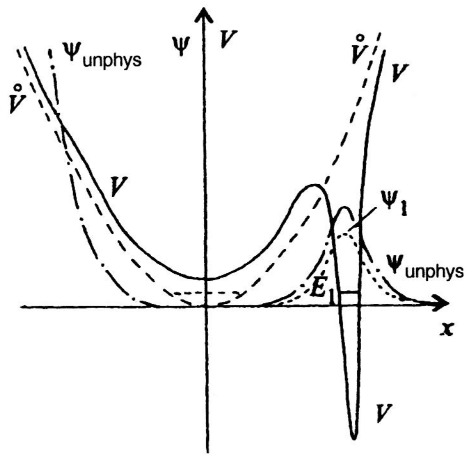

FIG. 11. The deformation of the oscillator potential well in the generation of an unphysical state with $E=E_{\rm unphys}$ below the ground bound state with $E=E_1$. The unphysical state was chosen to fall off to the right. This figure supplements that given in Ref. 18. In our case it is more obvious that the closer $E_{\rm unphys}$ is to E_1, the more similar to each other become the ground state $\psi_1(x)$ and the unphysical state $\psi_{\rm unphys}(x)$ on the right, moving off to the right in a typical carrier potential well [on the left, $\psi_{\rm unphys}(x)$ deviates from $\psi_1(x)$, growing exponentially]. The other levels of both the physical and the unphysical spectrum remain in place (on the energy scale) and are shifted relatively little on the x axis.

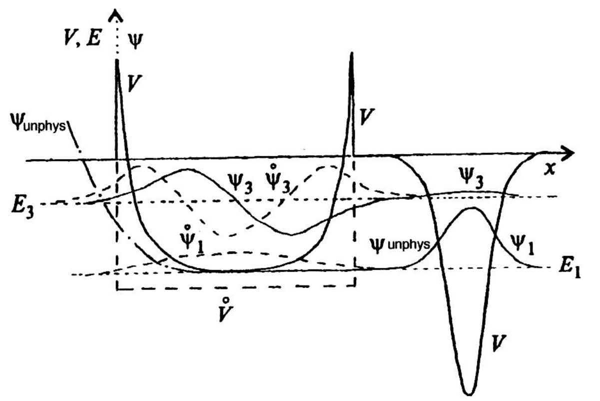

FIG. 12. The deformation of a square well of finite depth in the generation of an unphysical state with $E=E_{\rm unphys}$ below the ground bound state with $E=E_b$. As in the preceding examples, the unphysical state was chosen to fall off to the right.

still necessary to clearly formulate the rules governing the qualitative change of the spectral weight factors of all the physical states in this case.

What prevents two bound states from being concentrated in a small region of space?

In the sweeping of two square-well states to the coordinate origin, we discover the rudiments of the annihilation (repulsion) of states when they are degenerate.[17] The levels of the original, relatively wide well remain in place (they do not come closer together), but in the narrow well separated from the rest of space by the barrier causing the sweeping they are squeezed together. In a narrower well they would be expected to be farther apart in energy. This leads to the "breakup" of nearby states into two parts, as in the case of effective degeneracy (Fig. 13a). Therefore, only some of the functions remain on the left in the narrow well near $x=0$, while the rest are expelled to the outer region. Some of the decoupled functions become similar in shape (up to a sign), as in annihilation.[17]

A similar picture is also seen for two BSCs pushed toward the origin $x=0$ (see Fig. 13b). Comparing Figs. 5b and 13b, we see a common feature: the breakup of doublet states (in the first case as the states approach each other in energy, and in the second as they are pressed toward the coordinate origin). The difference is that in Fig. 5a the SWFs are several orders of magnitude smaller and the picture is less sharp at $x=0$ (the fragments into which the states break up contain very many antinodes, whereas in Fig. 13b each fragment contains one antinode of the wave function).

MULTICHANNEL SYSTEMS

Multichannel equations represent a powerful and universal tool for describing complex quantum objects. Systems of such equations form the basis of the Feshbach unified theory of quantum reactions and its generalization to processes involving particle redistribution,[33,34] which has made it more unified.[4)]

A multichannel system of coupled one-dimensional Schrödinger equations has the form

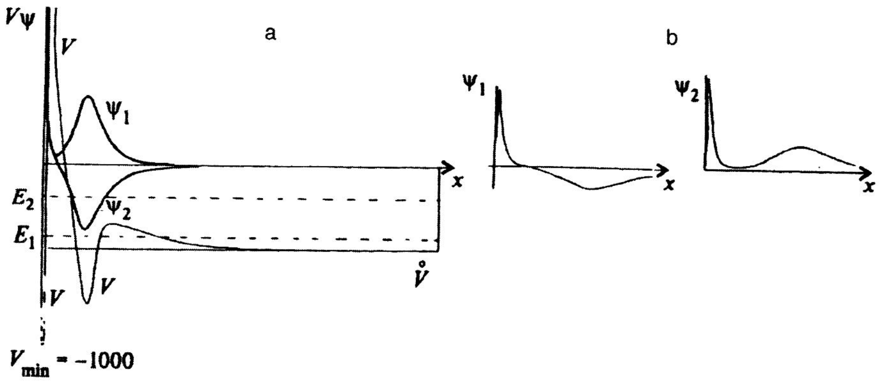

FIG. 13. (a) The "sweeping up" of two bound states with E_1, E_2 of the original square well as their SWFs are simultaneously increased. In the narrow, deep well formed at the coordinate origin by the high, narrow barrier in the potential V, the bound states Ψ_1, Ψ_2 come close together and are split as if they were degenerate[17] (effective degeneracy, even though in this case the levels do not approach each other). Antinodes of the same sign remain near $x=0$, and similar antinodes of the other sign are expelled beyond the barrier. (b) The sweeping up of two BSCs with $E_1=1$ and $E_2=4$ as the SWFs are increased to $C_1=C_2=1000$.

$$-\psi_i''(x)+\sum_j V_{ij}(x)\psi_j(x)=E_i\psi_i(x), \tag{21}$$

where, in contrast to the *scalar* one-channel case, the $\psi_j(x)$ are the components of a *vector* wave function, instead of the potential $V(x)$ we have the interaction *matrix* $V_{ij}(x)$, and the components $E_i=E-\epsilon_i$ of the energy vector are distinguished by the threshold energies ϵ_i. Instead of the scalar spectral weight factors (SWFs) c_λ, here the control levers are the spectral weight vectors (SWVs) with components $c_{\lambda,i}=\psi_i'(x=0)$, which enriches the possibility of transforming quantum systems.

Naturally, strong channel coupling allows the waves to flow easily from channel to channel, as in the overflow of a liquid into connected vessels.

In the one-channel case, as the SWF is increased the standing wave of a bound state is concentrated near the coordinate origin, and in the limit is pressed toward the origin by an infinite barrier.[8] It has been discovered[16] that for many channels in the $|c_i|\to\infty$ limit, for the ith partial wave all the components $\psi_{k\neq i,\lambda}(x)$ (and not only the ith component) are completely "pumped over" to the ith channel, where they are pressed toward the coordinate origin; see Fig. 14. Thus, the waves are "swept up" not only in configuration space, as in the one-channel case, but also in the discrete space of the channel variable i. Figure 14 demonstrates this intrinsically multichannel phenomenon for the example of two coupled channels on a bounded segment in x space between impenetrable potential walls.

The exact equations which we shall use here are given in Refs. 9, 16, and 35:

$$\psi_i(x,E)=\frac{\Sigma_j c_j\mathring{\Phi}_{ij}(x,E)}{1+\Sigma_k\int_0^r[\Sigma_j c_j\mathring{\Phi}_{kj}(y,E)]^2dy}, \tag{22}$$

$$\Phi_{ij}(x)=\frac{\mathring{\Phi}_{ij}(x,E)}{1+\Sigma_k\int_0^r[\Sigma_j c_j\mathring{\Phi}_{kj}(y,E)]^2dy}, \tag{23}$$

$$V_{ij}(x)=-2\frac{d}{dx}\frac{c_i\mathring{\Phi}_{ii}(x,E)c_j\mathring{\Phi}_{jj}(x,E)}{1+\Sigma_k\int_0^r[\Sigma_j c_j\mathring{\Phi}_{kj}(y,E)]^2dy}, \tag{24}$$

where $\mathring{\Phi}_{ij}(x,E)$ is the matrix of regular solutions of the Schrödinger equation with initial potential $\mathring{V}_{ij}(x)$ satisfying the boundary conditions at the coordinate origin:

$$\mathring{\Phi}_{ij}(0,E)=0,\quad \frac{d}{dx}\mathring{\Phi}_{ij}(x=0,E)=\delta_{ij}.$$

The levers of spatial shifts of individual states may be not only derivatives of the functions at the origin, $c_{\lambda,i}=\psi_i'(x=0)$. For example, in motion of the waves along the semiaxis $0\leqslant x\leqslant\infty$, in the inverse-problem approach it is possible to change, instead of the $c_{\lambda,i}$, the coefficients $M_{\lambda,i}$ of the decreasing exponentials in the asymptotic behavior of the bound-state waves in partial channels. In one-channel problems, increasing the coefficient M_λ is equivalent to decreasing c_λ, and in the limit $c_\lambda\to 0$ or $M_\lambda\to\infty$ the bound state with energy E_λ either is taken to infinity by the carrier well,[8] or is pressed toward the impenetrable potential wall limiting the wave motion as in an infinite square well. A special feature of the multichannel case is that in the limit $c_{\lambda,i}\to 0$ the corresponding partial wave does not vanish in the channel λ (it does not move off to infinity, and it is not pressed against any wall if there is one in its path), but, making use of the channel coupling, part of it moves into other channels and part moves toward the coordinate origin, from which it can later return to the original channel (see Fig. 15). Now the simultaneous vanishing of the function and

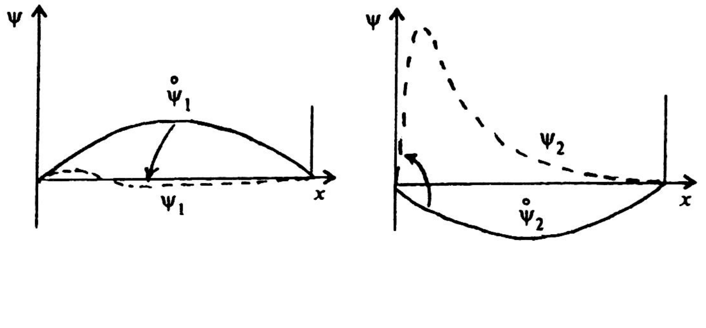

FIG. 14. Increase of the component c_2 of the SWV of the second channel for unchanged c_1 and the other spectral parameters: "pumping" of the waves from the first channel into the second in a two-channel system with "sweeping up" of the partial wave of the second channel toward the coordinate origin. The elements of the original interaction matrix are square wells.

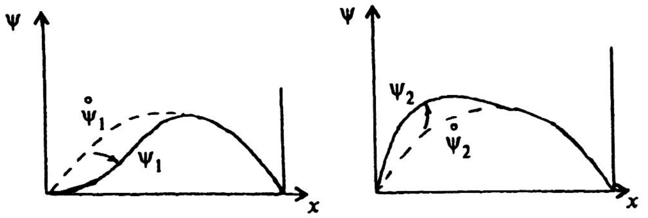

FIG. 15. The vanishing of the derivative of the wave function of the first channel at the coordinate origin, where $\psi_1'(0)=0$, does not cause $\psi_1(x)$ to vanish identically, as in the one-channel case. The function only partially goes into the second channel, from which it returns for $x>0$.

its first derivative at the same point do not cause the function to vanish over the entire range of the motion, as in the one-channel case, because it can be fed from other channels, owing to the coupling to them. If $M_{\lambda,i}\to\infty$, all the partial waves of the given state are concentrated in the ith channel and are carried to infinity by the carrier well (a "trap"), which is effectively equivalent to eliminating the bound state from the original system. For the original short-range Hamiltonian this carrier has the shape of a soliton,[5)] as in the one-channel case (see Fig. 16).

In the one-channel case the change of the SWF of a single bound state from one end of the integration range (accessible to the wave motion) does not change the other SWFs from the other end for the other bound states. In the multichannel case a change of the SWVs from one end changes all the SWVs from the other.

We expected that if the entire weight vector M were increased by scalar multiplication of it by a large number, a selected bound state would be removed from the original system in an individual transparent block[12] representing the multichannel analog of a soliton-like potential. This is true in the case of identical thresholds ϵ_i (see Fig. 17a). However, it turned out that in general the interaction matrix does not break up into two components between which $V_{ij}(x)$ practically vanishes and in which the spatially separated bound state and all the other bound states live. An example of such a separation of bound states without the vanishing of the elements of the interaction matrix in the intermediate region, as always occurs in the one-channel case, is given in Fig. 17b. In Fig. 17c we show the similar behavior of the interaction matrix when all the SWV components are decreased simultaneously.

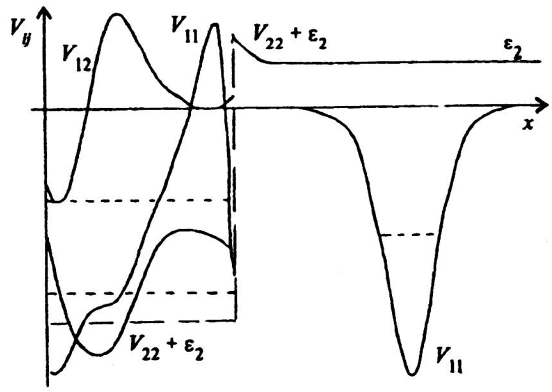

FIG. 16. As the coefficient M_1 in the exponentially falling tail of the first component of the second bound state is increased, the other components move into the first channel and are carried off in the soliton-like well $V_{11}(x)$. The elements of the original interaction matrix are square wells of finite depth. Cf. the case where the entire vector M in Fig. 17 is increased for identical and different thresholds.

We observed a similar unexpected behavior of the multichannel quantum system (with different thresholds ϵ_i) when the SWV c_λ was multiplied by a large scalar factor. Something prevents the vector bound state from simply being pressed toward the coordinate origin, as in the one-channel case. In Fig. 18 we show the "recoil" of the interaction matrix: on the side of increasing x, the block coupled to part of $V_{ij}(x)$ by relatively small but noticeable bridges is separated from it; cf. Fig. 17.

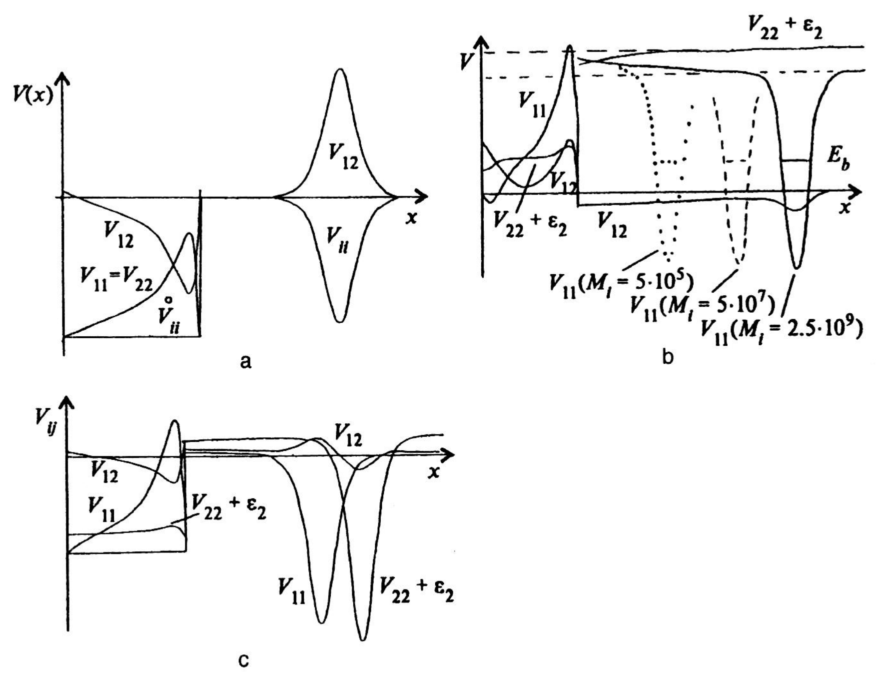

FIG. 17. The evolution of the original square-well interaction matrix as a single bound state is separated from the others by pulling or pushing (simultaneously increasing the weight factors of the asymptotic tails of the two components of the bound state or decreasing all the components of its SWV). (a) For identical thresholds this is accomplished by a carrier block of the interaction matrix with soliton-like matrix elements ($M_1=-M_2=10^6$). (b) For different thresholds, the bridges connecting the wells carrying the bound state to the rest of the interaction matrix do not vanish (there is no complete separation into two diverging blocks; $M_1=M_2=5\times10^5$, 5×10^7, and 2.5×10^9). (c) The same for the simultaneous decrease of $c_1=1.25\times10^{-5}$ and $c_2=10^{-5}$.

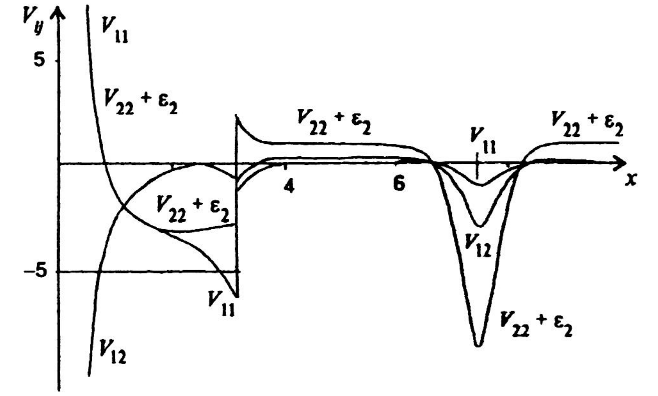

FIG. 18. The transformation of the original square-well two-channel interaction matrix when both components of the SWVs are increased simultaneously (sweeping them to the origin, $c_1 = c_2 = 10^4$). In a system with different thresholds, something causes the matrix $V_{ij}(x)$ to undergo a "recoil": on the right a block in which the function does not "live" is isolated. This block is connected to the main part of $V_{ij}(x)$ by bridges similar to those which appeared in Figs. 17b and 17c and in the separation of a single multichannel state with all the partial components nonzero.

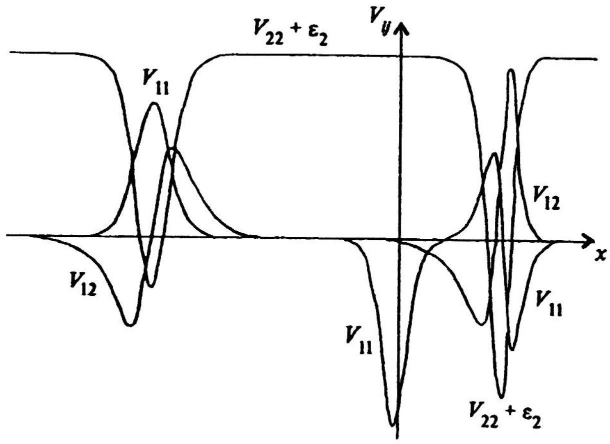

FIG. 20. The generation of two closely spaced ($E_1 = -0.5$ and $E_2 = -0.5001$) bound states with linearly independent SWVs. The original system corresponds to free motion in uncoupled channels with thresholds differing by ϵ. The transparent block not containing bound states is separated on the left.

When the SWV components are varied, nodes of the channel wave functions can appear and disappear; see Fig. 19 (in the one-channel case the nodes remain unchanged when the SWFs are varied).

We can also consider the transformation of multichannel states buried in the continuum. When only a single, the ith, component of the SWV $c_{\lambda,i} = \psi_i'(0,E_\lambda)$ is increased, all the other channel wave functions $\psi_{j\neq i}(x,E_\lambda)$ will also move into the ith channel. It was realized that it is possible to control the rate at which bound states buried in the continuum decay. Examples were given in Ref. 6 of exponentially decaying $V_{ij}(x)$ and $\psi_i(x)$ for BSCs below the threshold of the closed channel itself. It was later noticed that in special cases of BSCs between thresholds the falloff can be $\sim 1/x$. This is explained as follows. The matrix of regular solutions grows exponentially at large x. Solutions corresponding to scattering with energy between thresholds are constructed as linear combinations of columns of the matrix of regular solutions with coefficients ensuring exponential falloff in closed channels (without spoiling the ratios of the spectral weight components for the closed channels of the original physical scattering states). It turned out that the creation of BSCs with SWV components $c_{\lambda,i}$ corresponding to the same proportions (the "sweepup" of scattering states into BSCs with scalar change of the SWV components) leads to asymptotic falloff $\sim 1/x$ of $V_{ij}(x)$ and $\psi_{\lambda,i}(x)$. This is associated with linear growth of the integral in the denominator in equations like (24) due to nongrowing asymptotes of the scattering waves which move off to infinity. Any violation of these proportions does not lead to cancellation of the exponential growth in these denominators. Exponential falloff of $V_{ij}(x)$ and $\psi_{\lambda,i}(x)$ then results. It thereby becomes possible to control the rate of falloff of the potentials and functions corresponding to BSCs.

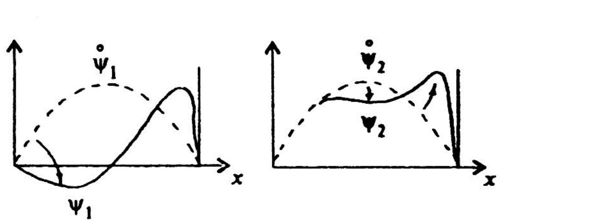

FIG. 19. The appearance of nodes in the partial channel functions in the evolution of a quantum system induced by changing the sign of the c_1 component of the SWV of the first channel; see also Fig. 14. In the one-channel case the number of nodes is conserved when the SWF is changed.

In the one-channel case we discovered[17] the phenomenon of the effective "annihilation" (breakup and repulsion) of bound states in the limit when the corresponding energy levels come close together. In the case of several (M) channels, it is possible for M degenerate states to coexist when they (or their SWVs) are linearly independent. Only linearly dependent states must experience repulsion.

However, we have seen that for channels with different thresholds, in the evolution of the interaction matrix $V_{ij}(x)$ as the energy levels of two bound states with independent SWVs come close together, spatial separation of some block of the interaction matrix occurs, as shown in Fig. 20. This block, which moves off to infinity in the degenerate limit, is transparent, which is reminiscent of one-channel annihilation,[16] but it does not carry off any bound state. All the bound states remain in the principal part of the interaction matrix. The explanation of this remains obscure. We shall discuss the properties of this block when we demonstrate its generation by means of the SUSYQ procedure.

Interestingly, the rudiments of the spatial breakup of functions (preparatory to "annihilation") can be observed not only when levels come close together, but also when the SWVs of two adjacent, closely spaced states with fixed levels come close together (with the levels remaining unchanged; see Fig. 21). We mentioned a similar phenomenon in Ref. 8, but at that time we did not know its correct explanation.

As in the one-channel case, states with linearly dependent SWVs will resist being swept to the origin (concentration in a narrow spatial region).

As two BSCs with identical SWVs come close together

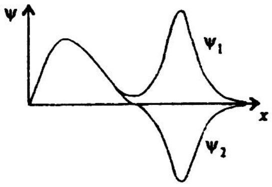

FIG. 21. The partial functions of one of the bound states, close in energy, of a two-channel system when the SWVs of the doublet come close together, but the level locations remain fixed. As the degree of linear dependence of the SWVs grows, the splitting of the states increases: the right-hand antinodes move farther and farther to the right.

in energy, splitting of the states occurs, as shown in Fig. 22 (cf. the phenomenon of effective annihilation discovered in the one-channel case[17]).

As in the one-channel case of a solution close in energy to a BSC with the same ratio of SWV components, beats are observed in the channel scattering functions, but with different beat frequencies.

Coexistence of bound states and scattering at the same energy

The "paradox" of the unification of opposite properties (the ability of an interaction to confine waves to form a bound state and at the same time to be transparent) and its resolution shed additional light on the fundamental features of multichannel quantum systems.

There should be no confusion between this case and the well studied quasibound states, which simultaneously possess the features of bound states (long delay and pileup of waves at the target) and scattering states.

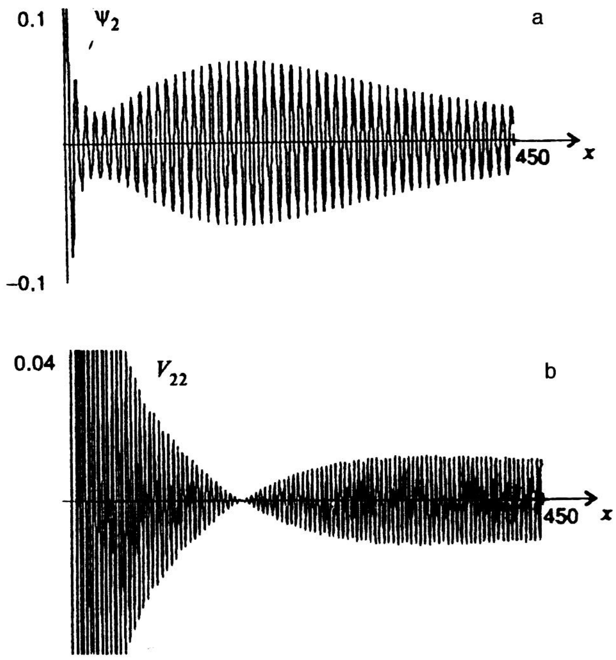

FIG. 22. The splitting of bound states buried in the continuum when they are generated with close energies. Only the individual components of the wave functions and interaction matrix are shown symbolically (the figure is compressed in the x direction in order to show the modulation of the oscillations of the functions and potentials).

There is no paradox if a spatially bounded region is isolated by an impenetrable (infinite) potential barrier, which leads to the appearance of (parallel) spectral branches: a discrete branch for bound states inside the constructed trap, and a continuous branch for scattering states outside the barrier. This is a trivial example of bound states buried in the continuum. There is no overlap of the regions of bound and scattering states, and the corresponding waves are in separate regions and experience different parts of the external field.

The same occurs for decoupled channels: the spectrum of the system is composed of the independent spectral branches of the individual channels. Then a bound state in one (for example, a closed) channel can coexist with scattering in a different channel not coupled to it at the same energy. This is not surprising, because again here states of different nature are formed by different interactions.

However, it is remarkable that even for strong coupling of open channels, special combinations of independent waves can get stuck in the interaction region (BSC confinement), while other waves can without any hindrance move into and out of the BSC localization region in the scattering process even without strict separation of the corresponding waves in configuration or channel space. We have shown[38] that the same interaction matrix is capable both of completely confining a wave and of having solutions corresponding to waves which move into and out of the interaction region; there can even be a transparent interaction, as will be explained below. The corresponding wave functions are obtained as various linear combinations of the set of independent solutions of the system of equations. In one combination the interchannel influence of the waves completely suppresses scattering, and in the other combinations it does not.

In the case of an original interaction matrix corresponding to a square well, the elements of the matrix of regular solutions $\Phi(x)$ have the form

$$\Phi_{lp}(x)=\frac{1}{k_l}\left(\sin(k_l x)\,\delta_{lp}-\frac{c_l c_p \sin(k_l x)\left(\frac{x}{2}-\frac{\sin(k_p x)}{4k_p}\right)}{1+c_1^2\left(\frac{x}{2}-\frac{\sin(k_1 x)}{4k_1}\right)+c_2^2\left(\frac{x}{2}-\frac{\sin(k_2 x)}{4k_2}\right)}\right), \tag{25}$$

where $k_i=\sqrt{E_i}$. These solutions contain both terms which fall off ($\sim 1/x$) and terms responsible for free wave propagation at asymptotic distances $x\to\infty$:

$$\Phi_{lp}(x)\to\frac{1}{k_l}\sin(k_l x)\,\delta_{lp}-\frac{c_l c_p}{c_1^2+c_2^2}\sin(k_l x)+O\left(\frac{1}{x}\right), \tag{26}$$

where the columns of the matrix of regular solutions $\Phi(x)$ become linearly dependent asymptotically and can be mutually annihilated in a combination corresponding to a BSC. In any other linear combination they give a scattering solution.

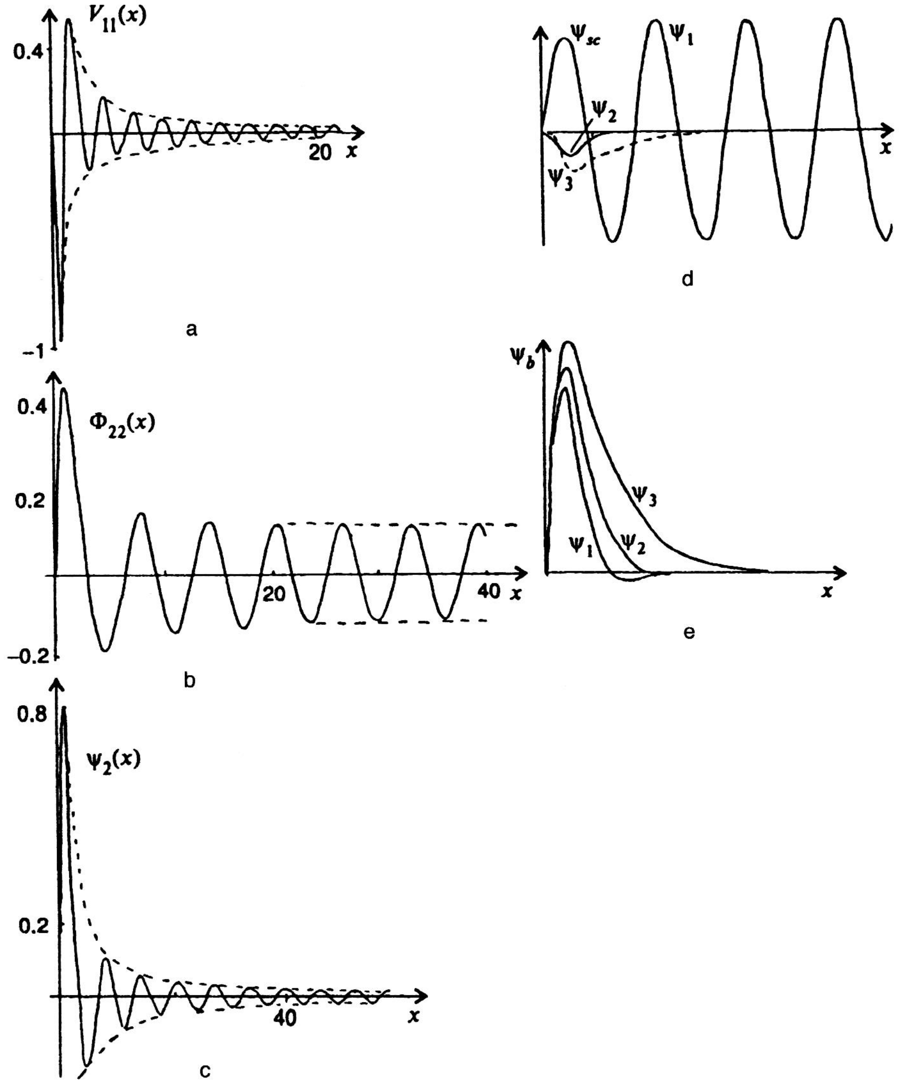

FIG. 23. Individual typical elements of the wave functions and interaction matrices in the case of the coexistence of bound states in the continuum with the same energy. (a, b, c) BSCs above the thresholds of the two-channel system; the oscillations of $V_{ij}(x)$, $\Psi_i(x)$, and $\Phi_{ij}(x)$ are damped as $1/x$. (d, e) BSCs of the three-channel system below the threshold of the third channel—the case where the BSC wave functions and potentials are exponentially damped in x.

If at a single energy as many independent BSCs (above all the thresholds) as there are channels in the system are created, then there will no longer be scattering at this energy (in the case of M channels there can be n bound and $M-n$ scattering states at the same energy). For three or more channels it is possible to construct bound states (BSCs) between the two upper thresholds which fall off exponentially with increasing x and coexist with scattering. This is achieved by violating the proportions in the linear combination of regular solutions for the original Hamiltonian, in which the growing components of the wave functions of closed channels cancel each other out. Then the presence of growing exponentials in the denominator of (27) ensures a short range of interaction and bound states. This cannot be achieved at an energy above all the thresholds, where only bound states falling off as $\sim 1/x$ can exist.[6)]

This limiting case for weak perturbation of the interaction matrix becomes the coexistence of two essentially different *eigen-channel* states at the same energy. One state is a resonance state with jump of the eigen-phase shift near the original BSC, and the other is a resonance-free state.

In Fig. 23 we show some of the typical elements (components) of the vector solutions for bound states, scattering, and also the interaction matrix

$$V_{ij}(x) = -2\frac{d}{dx}\frac{c_i c_j \frac{1}{k_i}\sin(k_i x)\frac{1}{k_j}\sin(k_j x)}{1+\Sigma_m \int_0^x c_m^2 \frac{1}{k_m^2}\sin(k_m y)^2 dy}. \tag{27}$$

The expressions for Φ_{ij} and V_{ij} can be checked by direct substitution into (21).

Therefore, in the one-channel case at the energy of a bound state buried in the continuum (in Neumann–Wigner potentials) there is no scattering, and multichannel bound and unbound states can coexist at the same spectral point.

Transparent systems

By analogy with the one-channel case, it can be expected that reflectionless interaction matrices are constructed from soliton-like potential wells. This is in fact true for identical channel thresholds. However, when the thresholds are different, in contrast to the one-channel case, where reflectionless potentials necessarily have a bound state, there are absolutely transparent multichannel interaction matrices with a purely continuous spectrum. They have been obtained by using the SUSYQ transformation.

The SUSYQ transformation in the multichannel case

Above, we considered the SUSYQ transformation for the one-channel case. Let us now give the multichannel generalization of this formalism.[16,19,21,22,26]

The Hamiltonian in (21) on the entire axis is written in factorized form:

$$\hat{H}_- = \hat{A}^+\hat{A}^- + \epsilon, \quad \epsilon < \epsilon_i, \tag{28}$$

where ϵ is the so-called factorization energy. The operator $\hat{A}^-$ becomes

$$\hat{A}^- = -\frac{d}{dx} + \hat{W}(x), \tag{29}$$

where $\hat{W}(x)$ is a matrix function whose form is, in general, not unique; its specific choice gives a definite potential transformation. The operator $\hat{A}^+ = d/dx + \{\hat{W}(x)\}^\dagger$ is the Hermitian conjugate of the operator $\hat{A}^-$. For definiteness, let us consider the case of only two channels, with the original potential matrix $V_{ij}(x)$ identically equal to zero on the entire axis. Let $\hat{\Psi}(x)$ be a (2×2)-matrix solution of the Schrödinger equation at energy ϵ [i.e., $\hat{H}_-\hat{\Psi}(x) = \epsilon\hat{\Psi}(x)$]. Each column of this matrix taken separately is a solution of the Schrödinger equation and can therefore be represented as a linear combination of, in general, four linearly independent solutions of the Schrödinger equation at a given energy. In our case (free motion) this means that each matrix element $\hat{\Psi}(x)$ can be represented as a combination of rising and falling exponentials:

$$\hat{\Psi}_{ij}(x) = \alpha_{ij}e^{-\kappa_i x} + \beta_{ij}e^{\kappa_i x}, \tag{30}$$

where the subscript i labels the rows of the matrix (i.e., the channels) and $\kappa_i = \sqrt{\epsilon_i - \epsilon}$. Thus, the choice of $\hat{\Psi}(x)$ in this case reduces to the choice of the coefficients α_{ij} and β_{ij}. In particular, we can take $\hat{\Psi}(x)$ to be of the form

$$\hat{\Psi}(x) = \begin{pmatrix} m_1 e^{-\kappa_1 x} & c_1 e^{\kappa_1 x} \\ m_2 e^{-\kappa_2 x} & c_2 e^{\kappa_2 x} \end{pmatrix}, \tag{31}$$

where we have set $\alpha_{11} \equiv m_1$, $\alpha_{12} = 0$, $\alpha_{21} \equiv m_2$, $\alpha_{22} = 0$, $\beta_{11} = 0$, $\beta_{12} \equiv c_1$, $\beta_{21} = 0$, and $\beta_{22} \equiv c_2$.

Now we can easily find $\hat{W}(x)$ from the equation

$$\hat{A}^- \hat{\Psi}(x) = 0, \tag{32}$$

which follows directly from the definition of the solution matrix $\hat{\Psi}(x)$ and the choice of the Hamiltonian (28). It is easy to see that

$$\hat{W}(x) = \hat{\Psi}'(x)\hat{\Psi}(x)^{-1}. \tag{33}$$

Using (31), we rewrite this expression as

$$\hat{W}(x) = \frac{1}{\det\hat{\Psi}(x)} \begin{pmatrix} -\kappa_1 m_1 e^{-\kappa_1 x} & \kappa_1 c_1 e^{\kappa_1 x} \\ -\kappa_2 m_2 e^{-\kappa_2 x} & \kappa_2 c_2 e^{\kappa_2 x} \end{pmatrix} \begin{pmatrix} c_2 e^{\kappa_2 x} & -c_1 e^{\kappa_1 x} \\ -m_2 e^{-\kappa_2 x} & m_1 e^{-\kappa_1 x} \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{-\kappa_1[m_1c_2e^{(\kappa_2-\kappa_1)x} + m_2c_1e^{(\kappa_1-\kappa_2)x}]}{m_1c_2e^{(\kappa_2-\kappa_1)x} - m_2c_1e^{(\kappa_1-\kappa_2)x}} & \dfrac{2\kappa_1c_1m_1}{m_1m_2e^{(\kappa_2-\kappa_1)x} - m_2c_1e^{(\kappa_1-\kappa_2)x}} \\ \dfrac{-2\kappa_2c_2m_2}{m_1c_2e^{(\kappa_2-\kappa_1)x} - m_2c_1e^{(\kappa_1-\kappa_2)x}} & \dfrac{\kappa_2[m_2c_1e^{(\kappa_1-\kappa_2)x} + m_1c_2e^{(\kappa_2-\kappa_1)x}]}{m_1c_2e^{(\kappa_2-\kappa_1)x} - m_2c_1e^{(\kappa_1-\kappa_2)x}} \end{pmatrix}. \tag{34}$$

The next step is to impose constraints on the coefficients in $\hat{\Psi}(x)$ such that the matrix $\hat{W}(x)$ is Hermitian (in our case, simply symmetric). Otherwise, factorization into the operator $\hat{A}^-$ in the form (29) and its Hermitian conjugate $\hat{A}^+$ will be possible only for a Hamiltonian including the operator d/dx, the first derivative with respect to the coordinate. In fact, let $\hat{W}$ not be a Hermitian matrix. Then

$$\hat{H}_- = \hat{A}^+\hat{A}^- + \epsilon = \left(\frac{d}{dx} + \{\hat{W}(x)\}^\dagger\right)\left(-\frac{d}{dx} + \hat{W}(x)\right) + \epsilon$$
$$= -\frac{d^2}{dx^2} + \hat{W}'(x) + \hat{W}(x)\frac{d}{dx} - \{\hat{W}(x)\}^\dagger\frac{d}{dx}$$
$$+ \{\hat{W}(x)\}^\dagger\hat{W}(x) + \epsilon. \tag{35}$$

The form (33) for the matrix W does not, in general, guarantee that it is Hermitian. However,

$$\hat{\Psi}^\dagger(\hat{W}^\dagger - \hat{W})\hat{\Psi} = \hat{\Psi}^\dagger\{\hat{\Psi}^{-1\dagger}\hat{\Psi}'^\dagger - \hat{\Psi}'\hat{\Psi}^{-1}\}\hat{\Psi}$$
$$= \hat{\Psi}'^\dagger\hat{\Psi} - \hat{\Psi}^\dagger\hat{\Psi}' = \hat{C},$$

where $\hat{C}$ is a constant matrix. The proof of the last statement is analogous to the derivation of the flux conservation law. We can also make this constant matrix vanish (and thereby make $\hat{W}$ a symmetric matrix) simply by the required specification of the parameters in (31). In our special case this leads to the condition $c_2 = -c_1 m_1\kappa_1/\kappa_2 m_2$. Then c_1 cancels in the numerator and denominator:

$$\hat{W}_{11}(x)=\frac{m_2^2\kappa_1\kappa_2\exp[(\kappa_1-\kappa_2)x]-m_1^2\kappa_1^2\exp[(\kappa_2-\kappa_1)x]}{m_1^2\kappa_1\exp[(\kappa_2-\kappa_1)x]+m_2^2\kappa_2\exp[(\kappa_1-\kappa_2)x]},$$

$$\hat{W}_{12}(x)=\hat{W}_{21}(x)=-\frac{2\kappa_1 m_1\kappa_2 m_2}{m_1^2\kappa_1\exp[(\kappa_2-\kappa_1)x]+m_2^2\kappa_2\exp[(\kappa_1-\kappa_2)x]},$$

$$\hat{W}_{22}(x)=\frac{m_1^2\kappa_1\kappa_2\exp[(\kappa_2-\kappa_1)x]-m_2^2\kappa_2^2\exp[(\kappa_1-\kappa_2)x]}{m_1^2\kappa_1\exp[(\kappa_2-\kappa_1)x]+m_2^2\kappa_2\exp[(\kappa_1-\kappa_2)x]}. \quad (36)$$

The SUSYQ transformation itself (or the Darboux transformation) consists of interchanging the operators $\hat{A}^-$ and $\hat{A}^+$ in (28). Thus, we obtain the transformed Hamiltonian (or the supersymmetric partner $\hat{H}_-$)

$$\hat{H}_+=\hat{A}^{-1}\hat{A}^++\epsilon=\hat{H}_--2\hat{W}'(x). \quad (37)$$

It is easy to see that solutions which are column vectors of the Schrödinger equation with $\hat{H}_+$ have the form[7)]

$$\hat{\Psi}_+(x,E)=\hat{A}^-\hat{\Psi}_-(x,E). \quad (38)$$

It is easily seen that the transformation (38) does not change the characteristics of the continuum: the matrix of reflection coefficients remains the same and identically equal to zero, i.e., there is transparency at all energies of the continuous spectrum. In contrast to the one-channel case, the matrix $\hat{\Psi}(x)^{-1}$ is no longer a solution of the Schrödinger equation for $E=\epsilon$. Nevertheless, new linearly independent solutions can be constructed at energy $E=\epsilon$ by acting, as in (38), with the operator $\hat{A}^-$ on the four linearly independent solutions of the Schrödinger equation for the original, unchanged Hamiltonian $\hat{H}_-$ at the same energy. However, it should be made clear that the operation $\hat{A}^-\hat{\Psi}_-(x,\epsilon)$ at first gives us only two linearly independent solutions, because the solutions $\hat{\Psi}_+(x,\epsilon)=\hat{A}^-\hat{\Psi}_-(x,\epsilon)$ are also solutions of the equation

$$\hat{A}^+\hat{\Psi}_+(x,\epsilon)=0, \quad (39)$$

since this equation vanishes identically if we use the definition of the matrix function $\hat{\Psi}_+(x,\epsilon)$ and the identity $\hat{A}^+\hat{A}^-=\hat{H}_--\epsilon$. However, this equation is symbolic notation for a system of homogeneous *first*-order differential equations, and so we have a fundamental system of two linearly independent solutions. We can immediately reduce the expressions for these two linearly independent solution columns $\hat{\Psi}_+(x,\epsilon)$ of the system (39), combining them to form the matrix $\hat{\Psi}_+(x,\epsilon)$ in order to compactify the notation:

$$\hat{\Psi}_+(x,\epsilon)=\{\hat{\Psi}(x)^{-1}\}^T, \quad (40)$$

where T denotes the transpose. In fact,

$$\hat{A}^+\{\hat{\Psi}(x)^{-1}\}^T=\left\{\frac{d}{dx}+\hat{W}(x)\right\}\{\hat{\Psi}(x)^{-1}\}^T$$
$$=-\{\hat{\Psi}(x)^{-1}\hat{\Psi}(x)'\hat{\Psi}(x)^{-1}\}^T+\hat{W}(x)\times\{\hat{\Psi}(x)^{-1}\}^T$$
$$=-\{\hat{\Psi}(x)'\hat{\Psi}(x)^{-1}\}^T\{\hat{\Psi}(x)^{-1}\}^T+\hat{W}(x)\times\{\hat{\Psi}(x)^{-1}\}^T$$
$$=-\{\hat{W}(x)\}^T\{\hat{\Psi}(x)^{-1}\}^T+\hat{W}(x)\times\{\hat{\Psi}(x)^{-1}\}^T=0, \quad (41)$$

where we have used the symmetry of the matrix $\hat{W}(x)$ and Eq. (33) and, in addition, the well known expression for differentiating an inverse matrix: $\{\hat{\Psi}(x)^{-1}\}'=-\{\hat{\Psi}(x)^{-1}\hat{\Psi}(x)'\hat{\Psi}(x)^{-1}\}$. Therefore, the only difference from the one-channel case, where the new solution at energy ϵ was obtained simply by inversion $\psi_+(x)=\psi(x)^{-1}$ of the original solution, is that an additional trivial procedure is performed.

The expression for the other linearly independent solutions $\hat{\Psi}_+^{\#}(x,\epsilon)$ written directly in the general matrix form is

$$\hat{\Psi}_+^{\#}(x,\epsilon)=\{\hat{\Psi}(x)^{-1}\}^T\int^x\{\hat{\Psi}(y)\}^T\hat{\Psi}(y)dy. \quad (42)$$

In fact, the second pair of linearly independent solutions, combined to form the matrix $\hat{\Psi}_+^{\#}(x,\epsilon)$, can be found as the solution of the inhomogeneous differential equation

$$\hat{A}^+\hat{\Psi}_+^{\#}(x,\epsilon)=\hat{\Psi}(x), \quad (43)$$

noting that if we act with the operator $\hat{A}^-$ on both sides of this equation, we obtain the expression $(\hat{H}_+-\epsilon)\hat{\Psi}_+^{\#}(x,\epsilon)=0$, i.e., these solutions satisfy the Schrödinger equation (21) with Hamiltonian $\hat{H}_+$. The solutions $\hat{\Psi}_+^{\#}(x,\epsilon)$ differ from $\hat{\Psi}_+(x,\epsilon)$ in that the former satisfy an inhomogeneous differential equation (more precisely, system of equations) (43), while $\hat{\Psi}_+(x,\epsilon)$ is the solution matrix of the system of homogeneous equations (39). This proves that the solutions $\hat{\Psi}_+^{\#}(x,\epsilon)$ are actually new linearly independent solutions of the Schrödinger equation. Finally, we shall seek a solution of (43) in the form

$$\hat{\Psi}_+^{\#}(x,\epsilon)=\{\hat{\Psi}(x)^{-1}\}^T\hat{C}(x), \quad (44)$$

i.e., by varying the multiplicative arbitrary "constant," as in the case of an ordinary, non-matrix, first-order differential equation. Substituting (44) into (43), we find $\hat{C}(x)$:

$$\hat{C}(x)=\int^x\{\hat{\Psi}(y)\}^T\hat{\Psi}(y)dy,$$

which proves (42). It is also easy to verify the correctness of (42) by directly substituting (42) into the Schrödinger equation (21) with the Hamiltonian H_+ (37).

It turns out that the choice of $\hat{\Psi}(x)$ in (31) corresponds to the case where it is impossible to construct a linear combination of the column-vector solutions which, like a bound state, is exponentially damped asymptotically. In other words, this transformation does not lead to the appearance of a bound state at energy ϵ. We therefore have a nontrivial two-channel potential transformation which does not change the continuum, but also does not generate a physical bound state.

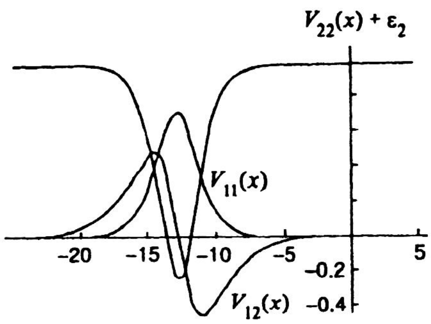

FIG. 24. An absolutely transparent two-channel interaction matrix having no analog in the one-channel case: it does not contain bound states. This potential block is isolated in $V_{ij}(x)$ in the generation of two bound states with similar energies; see Fig. 20. It enters as a component element into the more complicated two- and three-channel transparent matrices containing a bound state (see Fig. 25).

In Fig. 24 we show the transparent interaction matrix $V_{ij}(x)$ corresponding to (31) and (37). Interestingly, this interaction matrix appears as part of a transparent matrix with a bound state, discovered in Ref. 12 (see Fig. 18 in Ref. 1, in which the additional soliton-like well in the first channel on the right ensures the existence of a bound state in the system). Here, as in the transparent potential block with a bound state,[1,12,16] there is a potential barrier. Despite the fact that it leads to reflection, it is needed for total transmission—for the mutual suppression of these reflected waves and waves with the opposite phase coming from the second channel. In particular, it enters as a component into three- and higher-channel transparent matrices, as was first demonstrated in Ref. 12, as the elements of a "zipper"; see Fig. 25.

As noted above, a transparent block of this type without bound states is separated from the interaction matrix as the levels of two states with independent SWVs approach each other; see Fig. 20.

In the M-channel SUSYQ approach it is possible to *directly generate two (or even M) states* by means of a single, very simple transformation.

Actually, the example just considered is not the only case of all possible transformations determined by the choice of the coefficients α_{ij} and β_{ij}. For example, it is possible to take these coefficients [necessarily imposing a constraint related to the matrix symmetry of $\hat{W}(x)$ and inequality of the determinant of $\hat{\Psi}(x)$ to zero for all x] to all be nonzero and

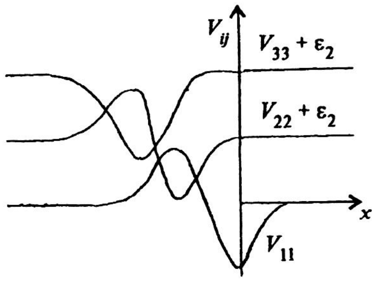

FIG. 25. The diagonal elements of an absolutely transparent three-channel interaction matrix with different thresholds and a bound state localized in the right-hand soliton-like well in $V_{11}(x)$. The elements similar to those in Fig. 24 should be noticed.

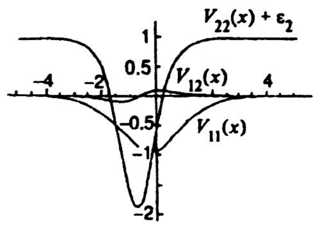

FIG. 26. An absolutely transparent interaction matrix with two degenerate bound states. It can be viewed as the limiting case in which a transparent block of the type shown in Fig. 24 was moved off to infinity, $x\to-\infty$. Figure 20 shows the intermediate case of incomplete degeneracy of the bound states, when a block not containing a bound state is moved off to infinity.

to have the same sign in each of the matrix elements of $\hat{\Psi}(x)$ (but this must be done in the entire matrix). After such a SUSYQ transformation we obtain a reflectionless system with two degenerate bound states at factorization energy ϵ; see Fig. 26.

In other words, by means of a single SUSYQ transformation we have generated *two* (!) bound states, although they are degenerate. It is easily verified that this is so by acting with the operator $\hat{A}^-$ on the four linearly independent solution columns of the unperturbed system of Schrödinger equations, which can be

$$\hat{\Psi}_-^{(1)}(x,\epsilon)=\begin{pmatrix} e^{\kappa_1 x} \\ 0 \end{pmatrix},\quad \hat{\Psi}_-^{(2)}(x,\epsilon)=\begin{pmatrix} e^{-\kappa_1 x} \\ 0 \end{pmatrix},$$

$$\hat{\Psi}_-^{(3)}(x,\epsilon)=\begin{pmatrix} 0 \\ e^{\kappa_2 x} \end{pmatrix},\quad \hat{\Psi}_-^{(4)}(x,\epsilon)=\begin{pmatrix} 0 \\ e^{-\kappa_2 x} \end{pmatrix}.$$

The new solutions $\hat{\Psi}_+^{(\nu)}(x,\epsilon)$, $\nu=1,\dots,4$, obtained from (38) all fall off exponentially at $\pm\infty$. Now they are not linearly independent solutions of the new system of Schrödinger equations. It can be shown that from the set $\hat{\Psi}_+^{(\nu)}(x,\epsilon)$ it is possible to select no more than two linearly independent solutions each time, from which it is possible to construct, by the appropriate linear superpositions, two mutually orthogonal solutions normalized to unity, i.e., wave functions of bound states degenerate in energy. It should be stressed that such degeneracy is possible in principle for multichannel systems only if the wave functions possess linearly independent spectral weight vectors, i.e., the factors (each for its own channel) multiplying the decreasing exponentials—the asymptotes of the channel wave functions of the bound states.

Let us consider yet another important feature of multichannel systems. It would appear that for interaction matrices $V_{ij}(x)$ which fall off rapidly asymptotically, the channels should actually be expected to decouple at large $|x|$, and in the case of different channel thresholds ϵ the solutions in the partial channels should become free solutions. However, the decreasing matrix elements coupling the channels can speed up the damping of the solutions in some channels by "pumping" the waves from these channels into other ones.

It therefore proved possible to invert the rate of exponential growth (decay) asymptotically for channel functions from opposite sides of the transparent block in the generation of a single bound state. This is possible for different thresholds ϵ. If on one side the modulus of the argument of the exponential is larger in a more closed channel (the normal situation), then the situation is the opposite on the other side. In the generation of two bound states, two such inversions restore the normal situation.

As already noted,[1] if a wave is incident on one side of a transparent block in some particular channel, then in the region where the interaction matrix is strong it will be split up among other channels, but subsequently it will be completely restored in the initial channel on the other side of the target. Even waves abandoned in open channels, where it would appear that nothing stops them from going off to infinity, must return to the channel from which they came (demonstrating exponential decay in an open channel not due to the wave passing through a barrier, but due to suction via the channel coupling, which itself falls off exponentially).

In the case of systems transparent on the entire x axis, including the limiting case of free motion, there is a "virtual level" at $E=0$ which becomes real for arbitrarily small additional attraction.

It should also be noted that it is possible for an arbitrary small addition of constant sign to the channel coupling ΔV_{12} to generate a bound state in the case where the original system corresponds to free motion in coupled channels or a transparent interaction matrix.

A transparent matrix without a bound state is characterized by energy parameters (of the unphysical state with given asymptotic growth generated by a SUSYQ transformation) ϵ, m_1, m_2.

Let us now obtain the expression for a double SUSYQ transformation. We subject a system which has already undergone a SUSYQ transformation [see Eq. (39) and the other equations associated with it] to another SUSYQ transformation according to the same scheme as above. For example, as $\Psi(x)$ we take a combination of the matrix solutions obtained in the first step:

$$\hat{\Psi}(x)=\{\Psi(x)^{-1}\}^T+c\{\Psi(x)^{-1}\}^T\int^x\{\Psi(y)\}^T\Psi(y)dy, \tag{45}$$

where c is an arbitrary constant. With this function, $\hat{V}(x)$ in the second step has the form

$$\begin{aligned}\hat{\hat{V}}(x)&=\hat{V}(x)-2\frac{d}{dx}\hat{\hat{W}}(x)=\mathring{V}(x)-2\frac{d}{dx}(\Psi'(x)\{\Psi(x)\}^{-1}\\&\quad+\hat{\Psi}'(x)\{\hat{\Psi}(x)\}^{-1})\\&=\mathring{V}(x)-2\frac{d}{dx}\left\{c\Psi(x)\right.\\&\quad\times\left.\left\{1+c\int^x[\Psi(y)]^T\Psi(y)dy\right\}^{-1}\Psi(x)^T\right\}.\end{aligned} \tag{46}$$

It is easily seen that with the choice

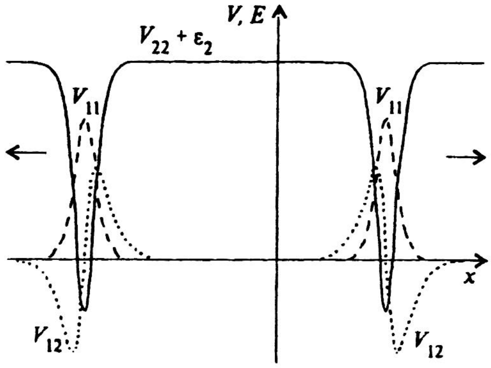

FIG. 27. The double SUSYQ transformation of a free two-channel system with close factorization energies corresponding to production of "unphysical" states. Repulsion of the transparent blocks of the interaction matrix is observed.

$$\Psi(x)=\begin{pmatrix}\frac{m_1}{\sqrt{c}}e^{-\kappa_1 x} & 0\\ \frac{m_2}{\sqrt{c}}e^{-\kappa_2 x} & 0\end{pmatrix},$$

Eq. (46) becomes exactly the expression $V_{ij}(x)=-2(d/dx)K_{ij}(x,x)$, where K is given by the equation obtained in the inverse-problem approach (level generation). Therefore, the inverse problem is a special case of (46), and the (single and double) supersymmetry transformation gives us a *larger class* of exactly solvable models, both for multichannel systems and for the case of the ordinary one-dimensional Schrödinger equation. We stress the fact that the formalism described above is also true in the general case of an arbitrary number of channels.

As in the one-channel case,[15] for spectrally equivalent SUSYQ transformations it is possible to have inversion of the interaction matrix (with the creation and annihilation of bound states).

Shortly before completing the work on this review, we generated two blocks corresponding to approaching factorization energies. It turned out that when these two blocks are combined, two parts moving away from each other appear, as shown in Fig. 27. This also is reminiscent of the repulsion of soliton wells as levels of physical solutions approach each other[17]—another surprise of multichannel systems.

In general, it is impossible to generate a bound state on the semiaxis in the Marchenko approach even in the one-channel case, because the requirement of not changing the phase shift $\delta(k)$ (the parameters of the continuum specified at infinity) contradicts the Levinson theorem, which requires that the phase shift at zero energy $\delta(0)$ change by π in this case. In other words, the creation of a level requires making the potential well deeper, which leads to a phase shift. However, if there is a centrifugal barrier with $l=2$ giving $\delta(0)=l\pi/2=\pi$, then the generation of a bound state removes this barrier, and $\delta(0)$ remains unchanged: the vanishing of the barrier decreases $\delta(0)$ by π, and the appearance of a bound-state level increases $\delta(0)$ by π, so that these changes cancel each other.[21] In the multichannel case, the generation of a level in the Marchenko approach requires a centrifugal barrier in at least one channel.

Interchannel motion

Owing to the possibility of wave motion between channels and not only along the spatial coordinate, a new resonance mechanism exists (a quasibound state is a standing wave in the channel variable).[36]

Interestingly, periodicity of the interaction matrix in the channel variable (for an infinite number of channels with identical thresholds) leads to a band spectrum (this was understood thanks to a question asked by a student called Yanchenko in a lecture at the Moscow Physico-Technical Institute).

Classical exactly solvable multiparticle models

In Ref. 37 (see also references therein) it was shown that the solution of the Schrödinger equation corresponds to the solution of a classical problem. We shall now use the multichannel quantum models that we have discovered by the approach of the inverse problem and SUSYQ to obtain, by analogy, free of charge, the corresponding broad classes of exactly solvable one-dimensional, but now multiparticle (!), classical models.

In the system (21) we replace the channel function $\psi_i(x)$ and its coordinate x by the coordinate of the ith classical particle and the time $x_i(t)$. Then the second derivative of $\psi_i(x)$ is replaced by the acceleration $\ddot{x}_i$, and the elements of the potential matrix $V_{ij}(x)$ cease to be potentials and become functions of time, i.e., they will be constituent forces depending both on coordinates and on time. The entire system (21) is transformed into a system of equations for several classical particles, where the particle accelerations will be determined by forces—the other terms in the equations:

$$\ddot{x}_i=\sum_j V_{ij}(t)x_j(t)-E_ix_i(t). \tag{47}$$

Here the parameters E_i no longer play the role of the channel energies, but simply enter into the definition of the forces acting in the system. Therefore, each exact solution of the multichannel quantum problem (and we have a complete set of them!) will correspond to an exact solution of the multiparticle classical problem with time-dependent forces. The asymptotic and boundary conditions of the quantum problems will determine the initial and final conditions of the corresponding classical solutions. Interestingly, for classical scattering solutions it is necessary to use unphysical quantum solutions which grow linearly asymptotically (for $E_i=0$). These results, obtained at the last moment, will be described in more detail in a special publication.

In this review we have focused on several new multichannel rules, but we still need to discover a number of other surprising phenomena before obtaining a complete picture of the qualitative transformations of complex systems. A solution of this very attractive problem will significantly enrich our quantum intuition.

[1)] It should be noted that supersymmetry in nonrelativistic quantum mechanics is understood as the reduction of the corresponding formalism for relativistic systems to one-dimensional space.

[2)] This theorem was proved by H. Borg for a finite interval, and generalized to infinite intervals by Gesztesy, Simon, and Teschl.[30] It would be interesting to find out whether or not it can be generalized to the case where the unphysical states are not normalizable and grow exponentially at large distances.

[3)] This does not occur when two unphysical states with identical boundary conditions approach each other, possibly because there is no orthogonality for such states.

[4)] The theory would become truly unified if it were supplemented by the formalism of the multichannel inverse problem, which is still in the developmental stage.[35]

[5)] This is true when there is a continuum and it is necessary to preserve the scattering features in the transformation of the bound state.

[6)] Between the upper thresholds there are $(M-1)$ independent solutions which do not have exponentially growing components in a closed channel. If only one channel is open, it is possible for only one physical state to exist at each value of the energy.

[7)] We do not give the proof, because it has exactly the same form as that for the one-channel SUSYQ transformation, which was already studied for the example of a periodic potential and on which there is already a very extensive bibliography.

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